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B. MATHEMATICS AND MATHEMATICAL PHYSICS

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Theory of Diffraction in Microwave Interferometry

D. M. Kerns and E. S. Dayhoff [1]

(August 25, 1959)

Microwave Michelson and Fabry-Perot interferometers are respectively considered as instances of: (1) A "reflection system", consisting of a radiating-receiving system and a reflecting object (e.g., a finite mirror); and (2) a "transmission system", consisting of a radiating system and a receiving system with an object (e.g., a Fabry-Perot etalon) interposed. The basic theoretical objective is the calculation of the amplitude and phase of the (time-harmonic) received signal in the systems considered. The electromagnetic field in space transmission paths is represented in terms of continuous angular spectra of vectorial plane waves, and the elements of the systems are described by means of suitable tensor scattering matrices (having both discrete and continuous indices). Needed scattering matrices are considered known; relationships to experimentally determinable data are outlined. The general case of either the reflection or transmission system is soluble formally in terms of a series of integrals stemming from the Liouville-Neumann series solution of certain integral equations. Formulas are obtained for models of the Michelson and Fabry-Perot instruments with arbitrary radiating and receiving characteristics. The theory and various features of the instruments considered, including Fresnel-region (or quasi-optical) behavior, are illustrated by means of examples obtained by choosing relatively simple and rather hypothetical analytical expressions for the radiating and receiving characteristics.

1. Introduction

Microwave versions of the Michelson and the Fabry-Perot interferometers are being studied and developed at the National Bureau of Standards with a view to high-accuracy measurements of wavelengths, lengths, and the speed of light [1].[2] The data provided by these instruments are, however, subject to corrections for effects which are completely negligible in optical interferometry but significant in the microwave region. In this work—in contrast to the corresponding optical case—the principal such effect results from the wavelength not being negligibly small relative to the dimensions of the apparatus, and consequently the effects of diffraction upon the wavelength as observed by the instrument must be carefully considered in order to exploit fully the potential accuracy of the method. In this paper we consider these effects and present analytical tools suitable for making a correction to the apparent wavelength [2]. This should make possible the use of Fresnel region microwave interferometry for work of the highest accuracy.

For the purposes of our analysis the pertinent aspects of the class of microwave Michelson interferometers are represented by a "general reflection system." Similarly, the microwave Fabry-Perot instruments are represented by a "general transmission system." (These basic arrangements are described more fully below.) Inasmuch as we wish to provide a theory inherently capable of dealing with high-accuracy experiments, a considerable degree of generality is required to avoid over-idealization. It is interesting that the required generality is such that the general reflection system could equally well represent a system consisting of radar and target, for example; similarly, the general transmission system could represent a point-to-point communication system. In view of the generality of the basic arrangements it seems likely that applications of the theory will also be found in problems other than the ones that motivated this work.

The specific theoretical objective is the calculation of the amplitude and phase of the received signal in the systems considered. Theoretical expressions for the received signal will contain, at least implicitly, all available information regarding diffraction errors, intensities, etc. The analytical technique employed involves a transducer point-of-view, scattering-matrix formalism and constitutes a generalization of a part of the theory of waveguides and waveguide junctions.

The arrangements to be considered involve radiating, receiving, and radiating-receiving systems as elements, which we shall occasionally refer to as "terminals" or as "terminal apparatus." Such an apparatus is shown schematically in figure 1. The left-hand block in the figure may contain sources of power, detectors, and any other required auxiliary equipment, all of which is assumed to be shielded. This equipment is connected by means of waveguide to a waveguide-space transducer, represented by the triangular symbol in the figure. For simplicity in the formulation it will be assumed that the waveguide is of a self-enclosed or shielded type. The arrangement and details of the apparatus are highly arbitrary, an essential but not very restrictive

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[2] Figures in brackets indicate the literature references at the end of this paper.

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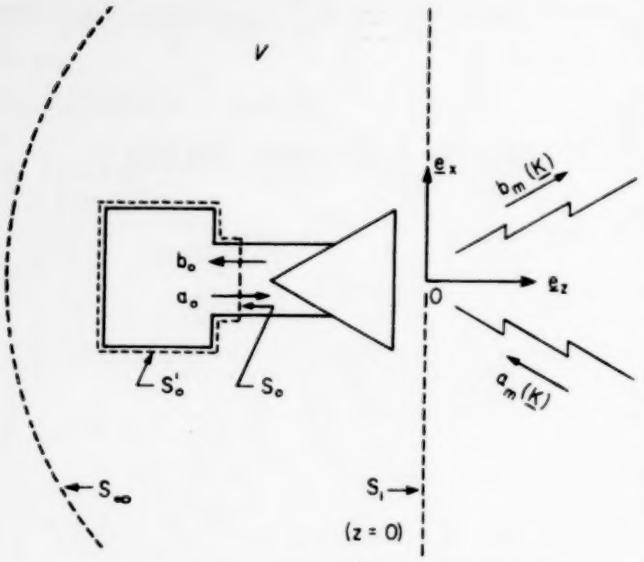

FIGURE 1. *Schematic radiating-receiving system.*

condition being that the transducer proper possesses the property of linearity with respect to electromagnetic fields. (Details of the statement of hypotheses are given in sec. 2, A.)

In one of the basic arrangements considered—the general reflection system, illustrated in figure 2—radiation from a radiating-receiving system is reflected or scattered by a general reflecting object at a variable distance d and is partly received (and partly reflected or scattered) by the same system. The Michelson interferometer can be considered in general terms as a reflection system in which the received signal is observed, as a function of d, by causing it to interfere with a suitable reference signal and detecting the resultant amplitude.

In the laboratory version of the Michelson interferometer being considered in connection with the present work, the radiation forms a well-defined beam (within its Fresnel zone), the reflecting object

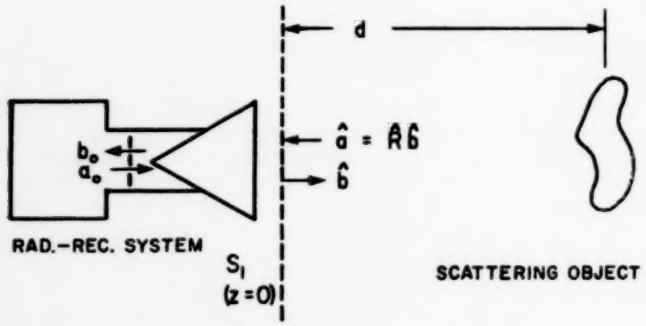

FIGURE 2. *Reflection system.*

If the scattering object is a finite mirror, for example, the arrangement represents a microwave Michelson interferometer.

is a large but finite mirror placed well within the Fresnel zone of the beam, and multiple reflections between the mirror and the radiating-receiving system are relatively unimportant. It may be said that the instrument is operated in the Fresnel or quasi-optical region. This mode of operation implies the inequalities $a \gg \lambda$ and $d \ll a^2/\lambda$, where a is a measure of the cross section of the beam and λ is the free-space wavelength corresponding to the frequency of operation. Under these conditions the radiation in the space between radiator and mirror approximates a homogeneous, plane standing-wave and the received signal is approximately proportional to exp $(2ikd)$, where $k = 2\pi/\lambda$.

The second basic arrangement considered is illustrated in figure 3. Here a structure or object that in general may reflect, transmit, and scatter is interposed between a radiating and a receiving system. If the interposed object is some form of Fabry-Perot "etalon" [3], the arrangement represents a general form of the Fabry-Perot interferometer. In this type of instrument the received signal exhibits amplitude variations sharply dependent on the spacing of the plates of the etalon, and this is attributable to interference among components of the radiation having experienced 0, 1, 2, . . . reflections between the plates.

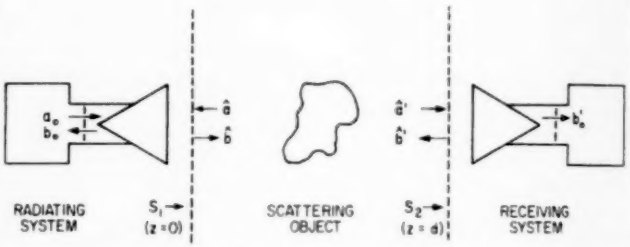

FIGURE 3. *Transmission system.*

If the scattering object is a suitable form of Fabry-Perot etalon, the arrangement represents a microwave version of the Fabry-Perot interferometer.

In the laboratory version of the Fabry-Perot being considered in connection with the present work, the instrument is a Fresnel-region instrument in the sense explained above, and multiple reflections, except between the plates of the etalon, are again relatively unimportant.

In the theoretical approach used in the present paper, the (vectorial) electromagnetic field in a space transmission path is represented as the superposition of an angular distribution or spectrum of (vectorial) plane waves. Such a spectrum is in general determined not only by the free-space radiation characteristics of a radiator, but also by the effects of all scattering elements involved (the radiator itself would be such an element, for example, in the case of a Michelson instrument with appreciable multiple reflections). Although the use of the plane-wave representation of the fields is suggested by the close approximation to a single plane wave that might exist in a Fresnel-region instrument, it is hardly

necessary to remark that in a practical case even the unmodified free-space radiation spectrum is likely to be extremely complicated in detail. This is because of such things as edge effects, geometrical imperfections, and limitations in the design and construction of lenses and horns.

The concept of the plane-wave resolution of the field permits one to form a useful qualitative picture of the origin of an effective wavelength. Consider, for example, a Michelson instrument. Each elementary wave in the spectrum whose normal makes an angle θ with the line along which translation of the mirror is measured has an effective wavelength $\lambda \sec \theta$; the resultant effective wavelength can be thought of as a kind of average of the elementary contributions over the existing spectrum. Thus one might anticipate, for example, that the resultant effective wavelength should rather generally tend to be greater than the free-space wavelength.

A main section of this paper, section 2, is devoted to establishing a scattering matrix formalism for the description of radiating-receiving systems of the general type considered. In this connection it is helpful—at least to one acquainted with the theory of waveguide junctions—to regard the space side of the waveguide-space transducer as a waveguide of infinite cross section. A scattering matrix having both discrete and continuous indices, which correspond respectively to the mode in the ordinary waveguide and the continuum of modes in the waveguide of infinite cross section, is required. The expression of reciprocity, which seems to be of interest in itself in the class of problems considered, is established.

For the purpose of the present work, the scattering matrices of the radiating-receiving systems involved are considered known; the determination of the necessary data for a practical application is considered to be an independent problem, experimental or possibly theoretical. (Fortunately in the practical cases under consideration, scattering by the radiating-receiving systems involved is of minor importance, and it thus appears that the necessary data can be obtained from suitable measurements directly or indirectly determining the far-field radiation characteristics. Such measurements would yield the necessary information about the underlying diffraction problem, however complicated it might be.)

In section 3 the scattering matrix formalism is extended and applied to the calculation of the received signal in the basic arrangements described above. The general problem involving multiple reflections is soluble formally (in the sense that the problem is reducible to quadratures) in the form of a series of integrals, which stem from the Liouville-Neumann series solution of a system of integral equations. Formulas are derived for models of the microwave Michelson and Fabry-Perot interferometers in which the radiating and receiving spectra are arbitrary. Finally, these formulas are illustrated by means of examples obtained by choosing specific and relatively simple analytical expressions for the spectra involved.

2. Scattering Matrix Description of Radiating-Receiving Systems

A. *Basic hypotheses.* To the general description of the radiating-receiving systems given in section 1 we add the following details.

We choose a terminal surface S_0 in the waveguide and a supplementary surface S_0' (such that S_0+S_0' forms a closed surface) coinciding with the shielding around source, receiver, etc. (See fig. 1.) We further choose a rectangular coordinate system Oxyz such that coordinate surface $z=0$, which we denote by S_1, will serve as a terminal surface on the space side of the waveguide-space transducer. As a convenient artifice we employ an infinite hemisphere S_∞ lying in $z \leq 0$ and centered at O. The interior—i.e., the domain of the electromagnetic field—of the transducer is the region V bounded externally by S_1+S_∞ and internally by S_0+S_0'. In figure 1 the structure of the transducer considered lies in V; other structures or objects may also be in V but no attempt is made to illustrate this possibility.

The whole of the space and structure within V is counted simply as a (decidedly) nonhomogeneous medium which may also be dissipative and anisotropic. Linearity is assumed and is essential; reciprocity, in the sense that tensors describing the medium within V are required to be symmetric, is assumed and is useful but not essential.

In the space transmission paths (i.e., in $z>0$) the medium is to be homogeneous, isotropic, and nondissipative, as well as linear. For the time being it is assumed that these properties hold for arbitrarily large z; the interposition of elements (such as a reflecting object) will be considered later. (The theory is not actually restricted to transmission media having the ideal characteristics listed, since, as will be seen, an arbitrary linear transmission medium is to be treated as a suitable interposed element.)

It is assumed that the electromagnetic field quantities vary harmonically with time t at frequency $\omega/(2\pi)$. We employ the usual complex electric and magnetic field vectors, **E, H,** which are functions of the position vector **r** of Oxyz, and omit the time dependent factor $\exp(-i\omega t)$.

It should be observed that in taking S_∞ to be hemispherical it is tacitly assumed that the problem is not two-dimensional (by a "two-dimensional" problem is meant one in which all quantities are independent of one rectangular coordinate perpendicular to the z-direction). To avoid unduly complicating the discussion, two-dimensional problems are not considered in the main part of this paper. However, two of the examples at the end of the paper are two-dimensional. For convenience in discussing these and other two-dimensional cases, some of the more important or less obvious of the needed formulas are summarized in an appendix (sec. 4, B).

B. *Representation of field on* S_0. It, as is assumed, only one mode (a propagated mode) is of importance

in the waveguide in the neighborhood of S_0, then from waveguide theory it is known that the transverse components $\mathbf{E}_t$, $\mathbf{H}_t$ of $\mathbf{E}$, $\mathbf{H}$ on S_0 may be written

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r})_t&=(a_0+b_0)\,\mathbf{e}(\mathbf{r}),\\ \mathbf{H}(\mathbf{r})_t&=(a_0-b_0)\,\mathbf{h}(\mathbf{r}),\end{aligned}\right\}(\mathbf{r}\text{ on }S_0) \tag{1}$$

where $\mathbf{e}$, $\mathbf{h}$ are real basis-fields for the mode involved. The basis-fields are subject to the impedance normalization

$$\mathbf{h}(\mathbf{r})=\eta_0\mathbf{n}\times\mathbf{e}(\mathbf{r}) \tag{2}$$

and the power normalization

$$\int_{S_0}\mathbf{e}(\mathbf{r})\times\mathbf{h}(\mathbf{r})\cdot\mathbf{n}dS=4\pi^2\eta_0, \tag{3}$$

where $\mathbf{n}$ is the unit normal on S_0 drawn into V and η_0 is the wave-admittance for the mode involved. These equations implicitly define the quantities a_0, b_0; it can be verified that a_0, b_0 so defined are respectively linear measures of the electric field of the incident and emergent traveling-wave components of the waveguide field at S_0. The net time-average power input P_0 at S_0 is given by

$$P_0=\mathrm{Re}\int_{S_0}\tfrac{1}{2}\mathbf{E}\times\overline{\mathbf{H}}\cdot\mathbf{n}dS=2\pi^2\eta_0(|a_0|^2-|b_0|^2), \tag{4}$$

where Re denotes that the real part is to be taken and the superposed bar denotes the complex conjugate.

C. *Representation of the field in the region* $z\geq 0$. As mentioned in the introduction, the electromagnetic field in the region $z\geq 0$ is to be represented as a superposition of plane-wave solutions of Maxwell's equations. This type of representation is well known, at least in the case of solutions of the scalar wave equation [4]; the generalization to the vector electromagnetic field offers no particular formal difficulty.

The electromagnetic field in the region under consideration satisfies Maxwell's equations in the form

$$\nabla\times\mathbf{E}=i\omega\mu\mathbf{H},\ \nabla\times\mathbf{H}=-i\omega\epsilon\mathbf{E}, \tag{5}$$

where μ, ϵ are constant real scalars representing respectively the permeability and the permittivity of the medium. (Rationalized MKS units are employed.) We derive our basis fields from the general plane wave

$$\left.\begin{aligned}\mathbf{E}&=\mathbf{T}\exp(i\mathbf{k}\cdot\mathbf{r}),\\ \mathbf{H}&=(\omega\mu)^{-1}\mathbf{k}\times\mathbf{T}\exp(i\mathbf{k}\cdot\mathbf{r}),\end{aligned}\right\} \tag{6}$$

which is a solution of (5) for any propagation vector $\mathbf{k}$ such that $k^2=\omega^2\mu\epsilon$ and any vector $\mathbf{T}$ satisfying the transversality relation $\mathbf{k}\cdot\mathbf{T}=0$. (In spite of this occurrence of "transversality," in what follows the term "transverse" will always mean transverse with respect to the z-direction.)

The propagation vector will be regarded as a function of its transverse components k_x, k_y; the z-component is then

$$k_z=\pm\gamma, \tag{7a}$$

where $\gamma=(k^2-k_x^2-k_y^2)^{\frac{1}{2}}$. It will be convenient to denote the transverse part of the propagation vector by $\mathbf{K}$, so that $\mathbf{K}=k_x\mathbf{e}_x+k_y\mathbf{e}_y$ and

$$\gamma=(k^2-K^2)^{\frac{1}{2}}. \tag{7b}$$

Since k_x, k_y must be allowed to vary independently in the range $(-\infty,\infty)$, real and imaginary values of γ will occur. γ will be taken positive for $K^2<k^2$, positive imaginary for $K^2>k^2$. Superscripts "+" and "−" will be used when it is desired to indicate the choice of sign associated with k_z.

In virtue of the relation $\mathbf{k}\cdot\mathbf{T}=0$, (6) yields just two linearly independent fields, hence just two basis fields, for any given $\mathbf{k}$. The appropriate polarizations for the basis fields are those with the electric vectors parallel or perpendicular to the plane of $\mathbf{k}$ and $\mathbf{e}_z$, which is the plane of incidence for a ray incident on any plane $z=$const. This choice of polarizations corresponds to the practice in electromagnetic theory in deriving the Fresnel equations, for example; it also corresponds to the resolution into "transverse magnetic" and "transverse electric" modes of waveguide theory.

In setting up the desired basis fields it is convenient to employ the transverse unit vectors

$$\boldsymbol{\kappa}_1=\mathbf{K}/K,\ \boldsymbol{\kappa}_2=\mathbf{e}_z\times\boldsymbol{\kappa}_1, \tag{8}$$

which are respectively parallel and perpendicular to the plane of $\mathbf{k}$ and $\mathbf{e}_z$. (The notation is illustrated in fig. 4.) As a temporary abbreviation we put $u^{\pm}=\exp(i\mathbf{k}^{\pm}\cdot\mathbf{r})$. For the "$E_\parallel$" (or TM) components we take $\mathbf{T}=\boldsymbol{\kappa}_1\mp K\gamma^{-1}\mathbf{e}_z$ and obtain from (6)

$$\left.\begin{aligned}\mathbf{E}_1^{\pm}&=[\boldsymbol{\kappa}_1\mp K\gamma^{-1}\mathbf{e}_z]u^{\pm},\\ \mathbf{H}_1^{\pm}&=\pm\eta_1\mathbf{e}_z\times\boldsymbol{\kappa}_1u^{\pm},\end{aligned}\right\} \tag{9}$$

where $\eta_1=\omega\epsilon/\gamma$. For the "$E_\perp$" (or TE) components we take $\mathbf{T}=\boldsymbol{\kappa}_2$ and obtain from (6)

$$\left.\begin{aligned}\mathbf{E}_2^{\pm}&=\boldsymbol{\kappa}_2u^{\pm}\\ \mathbf{H}_2^{\pm}&=[\pm\eta_2\mathbf{e}_z\times\boldsymbol{\kappa}_2+K(\omega\mu)^{-1}\mathbf{e}_z]u^{\pm}\end{aligned}\right\} \tag{10}$$

where $\eta_2=\gamma/(\omega\mu)$. Among other similarities it may be observed that η_1, η_2 are "wave-admittances" that correspond exactly to the wave-admittances encountered in the theory of waveguides with discrete modes. The expressions in (9) and (10) are essen-

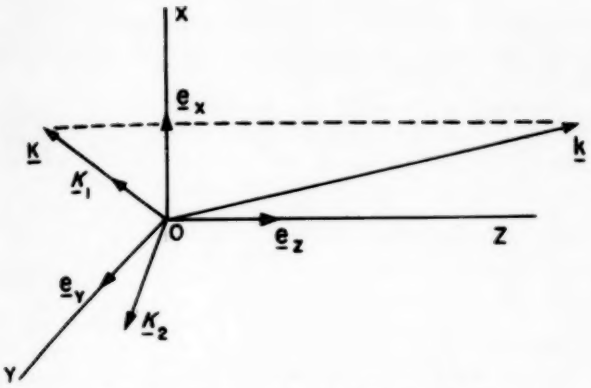

FIGURE 4. *Illustrating* $\mathbf{k}$, $\mathbf{K}$, κ_1, *and* κ_2.

tially the desired basis fields (z-components, included here, will be dropped later). The normalization and orthogonality properties of these fields are of course implicit in the expressions themselves.

We now introduce spectral density functions $b_m=b_m(\mathbf{K})$ and $a_m=a_m(\mathbf{K})$ for outgoing and incoming waves, respectively, and form the general superposition

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r})&=\int\sum(b_m\mathbf{E}_m^+ + a_m\mathbf{E}_m^-)d\mathbf{K},\\ \mathbf{H}(\mathbf{r})&=\int\sum(b_m\mathbf{H}_m^+ + a_m\mathbf{H}_m^-)d\mathbf{K}.\end{aligned}\right\} \tag{11}$$

Here and subsequently in expressions of this type summation over the two values of the polarization index m and integration over the domain of the two variables k_x, k_y will be understood. The $\mathbf{E}$, $\mathbf{H}$ given by (11) will satisfy Maxwell's equations provided merely that the necessary differentiations can be taken under the integral sign. However, for our purposes, the z-components of the fields are redundant. By discarding the z-components we obtain the much more convenient and explicit expressions for the transverse field components

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r})_t&=\int\sum(b_m e^{i\gamma z}+a_m e^{-i\gamma z})\boldsymbol{\kappa}_m e^{i\mathbf{K}\cdot\mathbf{R}}d\mathbf{K},\\ \mathbf{H}(\mathbf{r})_t&=\int\sum(b_m e^{i\gamma z}-a_m e^{-i\gamma z})\eta_m\mathbf{e}_z\times\boldsymbol{\kappa}_m e^{i\mathbf{K}\cdot\mathbf{R}}d\mathbf{K},\end{aligned}\right\} \tag{12}$$

where $\mathbf{R}$ denotes the transverse part of $\mathbf{r}$. These equations exhibit $\mathbf{E}_t$, $\mathbf{H}_t$ as two-dimensional Fourier transforms of the quantities multiplied by exp $(i\mathbf{K}\cdot\mathbf{R})$ in the respective integrands; from the inverse transformations one may obtain

$$\left.\begin{aligned}b_m(\mathbf{K})&=\frac{e^{-i\gamma z}}{8\pi^2}\boldsymbol{\kappa}_m\cdot\int(\mathbf{E}+\eta_m^{-1}\mathbf{H}\times\mathbf{e}_z)e^{-i\mathbf{K}\cdot\mathbf{R}}d\mathbf{R},\\ a_m(\mathbf{K})&=\frac{e^{i\gamma z}}{8\pi^2}\boldsymbol{\kappa}_m\cdot\int(\mathbf{E}-\eta_m^{-1}\mathbf{H}\times\mathbf{e}_z)e^{-i\mathbf{K}\cdot\mathbf{R}}d\mathbf{R}.\end{aligned}\right\} \tag{13}$$

Here and subsequently integrations with the differential symbol $d\mathbf{R}$ are to be taken over all values of x and y for some fixed $z\geq 0$. These equations enable one in principle to evaluate a_m, b_m from a given distribution of $\mathbf{E}_t$, $\mathbf{H}_t$ on any transverse plane.

The case where there is no field incident on S_1 from the "right" [i.e., where $a_m(\mathbf{K})\equiv 0$] is of particular interest in what follows, and we shall list several relations holding in this case. From (12) and (13)

$$\mathbf{E}(\mathbf{r})_t=\int\mathbf{B}(\mathbf{K})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{K}, \tag{14}$$

$$\mathbf{B}(\mathbf{K})=(2\pi)^{-2}e^{-i\gamma z}\int\mathbf{E}(\mathbf{r})_t e^{-i\mathbf{K}\cdot\mathbf{R}}d\mathbf{R} \tag{15}$$

where $\mathbf{B}=\mathbf{B}(\mathbf{K})$ is the transverse vector $b_1\boldsymbol{\kappa}_1+b_2\boldsymbol{\kappa}_2$. The time-average power, P_r, radiated into the half-space $z>0$ will also be of interest. This power is the same as the time-average power flux in the $+z$-direction across the surface S_1. Hence, from (12) [with $a_m(\mathbf{K})\equiv 0$],

$$P_r=\frac{1}{2}\mathrm{Re}\int\mathbf{E}\times\overline{\mathbf{H}}\cdot\mathbf{e}_z d\mathbf{R}=2\pi^2\int_{K^2<k^2}\sum\eta_m(\mathbf{K})|b_m(\mathbf{K})|^2 d\mathbf{K}, \tag{16}$$

where, as indicated by the notation below the integral sign, evanescent waves are excluded from the integration. This equation is of a type known as Parseval's formulas; it may be derived formally with the aid of the rule that

$$\int_{-\infty}^{\infty}e^{i(k_x-k_x')x}dx=2\pi\delta(k_x-k_x'), \tag{17}$$

where δ denotes the so-called impulse or δ-function [5].

D. *Definition of scattering matrix.* Having set up representations for the fields on the two terminal surfaces of the waveguide-space transducer under consideration, we are now in a position to consider the transducer as a whole. It may be assumed that a set of out-going wave-amplitudes [b_0 and $b_m(\mathbf{K})$] will be determined by a set of incident wave-amplitudes [a_0 and $a_m(\mathbf{K})$]. In fact, since the electromagnetic system under consideration is by hypothesis a linear system, the relation between the set of out-going wave-amplitudes and the set of incident amplitudes must be a linear relation. We write this

$$b_0=S_{00}a_0+\int\sum S_{01}(m,\mathbf{K})a_m(\mathbf{K})d\mathbf{K} \tag{18a}$$

$$b_m(\mathbf{K})=S_{10}(m,\mathbf{K})a_0+\int\sum S_{11}(m,\mathbf{K};n,\mathbf{L})a_n(\mathbf{L})d\mathbf{L}, \tag{18b}$$

thereby defining the scattering matrix for the transducer considered. Figure 1 may be helpful in fixing the significance of the quantities involved in (18). It is convenient and it seems appropriate to use the

term "matrix" here even though one must think of rows and columns labeled both by discrete indices and by indices having continuous ranges. Evidently the functions S_{01}, S_{10}, and S_{11} respectively embody the receiving properties, the radiating properties, and the (space-side) scattering properties of the transducer involved; S_{00} is an ordinary waveguide reflection coefficient defined at the terminal surface S_0 and expressing the "antenna mismatch."

Equation (18) can be represented in terms of S_{00} and linear functional operators $\hat{S}_{01}$, $\hat{S}_{10}$, and $\hat{S}_{11}$ such that

$$b_0 = S_{00}a_0 + \hat{S}_{01}\hat{a}, \tag{19a}$$

$$\hat{b} = \hat{S}_{10}a_0 + \hat{S}_{11}\hat{a}, \tag{19b}$$

where $\hat{b}$ and $\hat{a}$ are understood as function vectors corresponding to the functions $b_m(\mathbf{K})$ and $a_m(\mathbf{K})$. This compact notation is used later primarily as a convenience in some of the more formal and general parts of the discussion.

E. *Reciprocity.* Radiating and receiving characteristics are related by the reciprocity condition, which here takes the form

$$-\eta_0 S_{01}(m, \mathbf{K}) = \eta_m(\mathbf{K}) S_{10}(m, -\mathbf{K}). \tag{20}$$

Since this particular form of the condition appears to be new, a derivation is given below (appendix). The occurrence of the factors η_0 and $\eta_m(\mathbf{K})$ in (20) can be regarded as a consequence of the particular normalizations adopted in setting up the basis fields. (The same type of relationship holds for the elements of the scattering matrix of an ordinary waveguide junction with discrete modes [6].) The occurrence of the argument $-\mathbf{K}$ in one side of (20) means that the equation relates radiating and receiving characteristics in the line of a given propagation vector $\mathbf{k}$ (if a radiated wave has propagation vector $\mathbf{k}$, the received wave in the same line has the propagation vector $-\mathbf{k}$).

The plane-wave into plane-wave scattering function S_{11} is also subject to reciprocity; the relation is

$$\eta_m(\mathbf{K}) S_{11}(m, -\mathbf{K};\ n, \mathbf{L}) = \eta_n(\mathbf{L}) S_{11}(n, -\mathbf{L};\ m, \mathbf{K}). \tag{21}$$

Comments similar to those following (20) apply here also. This relation is not used in the present paper but it may well be of interest in other diffraction problems.

F. *Determination of scattering matrix.* Although the basic viewpoint of this paper is that S_{00}, S_{01}, etc., are to be considered known, it seems well to take some note of the problem of determining these quantities from empirical data in the arrangements of particular interest.

Concerning S_{00} and S_{11} there is not much to be said. S_{00} is not only relatively easily measurable but also experimentally controllable (by means of tuning elements), whereas in general the same is not at all true of S_{11}. However, the desired operating condition that S_{11} be effectively negligible is approximately attainable and to some extent subject to experimental verification (by observation of the effects of multiple reflections).

Concerning S_{01} and S_{10} we first note: (1) Either one of these functions may be determined readily from the other with the aid of the reciprocity relation; (2) in the present context these functions need be evaluated only for $K^2 < k^2$, the effects of evanescent waves being avoided by keeping $d >> \lambda$ (this is not inconsistent with Fresnel-region operation, cf sec. 1).

A direct approach to the determination of S_{01} is implied by the definition (18): S_{01} represents the received signal b_0 as a function of the direction and the polarization of incident plane waves of suitably normalized amplitudes.

According to (18), $S_{10}(m, \mathbf{K})$ represents the transverse components of the vector spectrum of outgoing waves under the conditions $a_m(\mathbf{K}) \equiv 0$ and $a_0 = 1$. If we define the transverse vectorial spectrum

$$\mathbf{S}_{10}(\mathbf{K}) = S_{10}(1, \mathbf{K})\boldsymbol{\kappa}_1 + S_{10}(2, \mathbf{K})\boldsymbol{\kappa}_2 \tag{22}$$

then, from (15),

$$\mathbf{S}_{10}(\mathbf{K}) = \frac{1}{4\pi^2 a_0} \int \mathbf{E}_t(\mathbf{R}) e^{-i\mathbf{K}\cdot\mathbf{R}} d\mathbf{R}; \tag{23}$$

that is, $\mathbf{S}_{10}$ may be represented as the Fourier transform of the transverse components of the electric field obtaining on the reference plane in the absence of incident waves, normalized to unit a_0. This is, of course, essentially a vector form of the well-known relation between "aperture" distribution and spectrum.

The vectorial spectrum $\mathbf{S}_{10}(\mathbf{K})$ is also closely related to the far electric field by a well-known type of relation. Under certain restrictions $\mathbf{E}(\mathbf{r})_t$ has for large r the asymptotic form [7]

$$\mathbf{E}(\mathbf{r})_{t,\,\mathrm{asymp}} = -2\pi i k \cos\theta\, \mathbf{B}(\mathbf{R}k/r) e^{ikr}/r; \tag{24}$$

the angle θ introduced here is the polar angle of $\mathbf{r}$ relative to the z-axis. By rewriting this equation and dividing by a_0 we obtain a formula for $\mathbf{S}_{10}(\mathbf{K})$ in terms of the asymptotic form of $\mathbf{E}_t$:

$$\mathbf{S}_{10}(\mathbf{K}) = i(2\pi a_0)^{-1}\gamma^{-1} r\, e^{-ikr} \mathbf{E}(\mathbf{k}r/k)_{t,\,\mathrm{asymp}}, \tag{25}$$

for $K^2 < k^2$.

Finally we note that the familiar "power radiation pattern" or "polar diagram" of antenna theory, defined as radiated power per unit solid angle as a function of direction, is given by

$$p = 2(\epsilon/\mu)^{\frac{1}{2}} (\pi k \cos\theta)^2\, \mathbf{b}\cdot\bar{\mathbf{b}}, \tag{26}$$

where $\mathbf{b} = \mathbf{B} + b_z\mathbf{e}_z$ and $b_z = -\mathbf{B}\cdot\mathbf{K}/\gamma$ ($\mathbf{b}$ is the complete vectorial angular spectrum, including the z-component). Clearly this equation is not sufficient by itself to determine $\mathbf{S}_{10}(\mathbf{K})$; polarization and phase information is required in addition.

3. Applications

A. *Reflection systems.* One obtains a form of the first basic arrangement described in section 1 by placing an infinite plane reflecting surface "in front of" a radiating-receiving system of the type considered in section 2. This represents a problem of intermediate complexity, from which the basic equations for the Michelson arrangement may be obtained by specialization. If the reflecting surface is at $z=d$ and has reflection coefficient $\rho(m,\mathbf{K})$, then, transforming the reflection coefficient to the plane $z=0$, we have

$$a_m(\mathbf{K})=\rho(m,\mathbf{K})e^{2i\gamma(\mathbf{K})d}b_m(\mathbf{K}) \tag{27}$$

(the dependence of γ on its arguments is now indicated explicitly). Upon substituting (27) into (18) one obtains

$$b_0=S_{00}a_0+\int\sum S_{01}(m,\mathbf{K})\,\rho(m,\mathbf{K})e^{2i\gamma(\mathbf{K})d}b_m(\mathbf{K})d\mathbf{K}, \tag{28a}$$

$$b_m(\mathbf{K})=S_{10}(m,\mathbf{K})a_0 + \int\sum S_{11}(m,\mathbf{K};n,\mathbf{L})\,\rho(n,\mathbf{L})e^{2i\gamma(\mathbf{L})d}b_n(\mathbf{L})d\mathbf{L}. \tag{28b}$$

The last line represents two simultaneous, inhomogeneous, linear integral equations for the determination of $b_m(\mathbf{K})$ (a_0 being prescribed). The Liouville-Neumann series solution of these equations may be obtained by a process of successive approximations. For the first approximation one takes

$$b_m^{(1)}(\mathbf{K})=S_{10}(m,\mathbf{K})a_0; \tag{29}$$

the second approximation is obtained by substituting the first into the right-hand side of (28b),

$$b_m^{(2)}(\mathbf{K})=S_{10}(m,\mathbf{K})a_0 + a_0\int\sum S_{11}(m,\mathbf{K};n,\mathbf{L})\,\rho(n,\mathbf{L})e^{2i\gamma(\mathbf{L})d}S_{10}(n,\mathbf{L})d\mathbf{L}; \tag{30}$$

and so on: $b_m^{(n)}(\mathbf{K})$ accounts for the first n reflections at the reflecting surface. Once $b_m(\mathbf{K})$ is obtained, approximately or otherwise, it is to be substituted into (28a), thus determining the received wave-amplitude b_0 in the waveguide at S_0.

Useful approximate equations describing the behavior of the Michelson instrument may now be obtained. The appropriate conditions are (1) that the effects of multiple reflections be negligible and (2) that the reflecting surface at $z=d$ be a mirror, for which we may put $\rho=-1$. The first condition means that (29) is already a good approximation for $b_m(\mathbf{K})$; this substituted into (28a) yields

$$\frac{b_0}{a_0}=S_{00}-\int\sum S_{01}(m,\mathbf{K})e^{2i\gamma(\mathbf{K})d}S_{10}(m,\mathbf{K})d\mathbf{K}. \tag{31}$$

The second term on the right will be called the "reflection integral" and denoted by $\Phi(d)$. Either S_{01} or S_{10} may be eliminated by means of the reciprocity condition (20); for the purposes of the present discussion it seems preferable to eliminate S_{01}. We then have

$$\Phi(\mathrm{d})=\eta_0^{-1}\int\sum \eta_m(\mathbf{K})\,S_{10}(m,\mathbf{K})\,S_{10}(m,-\mathbf{K})\,e^{2i\gamma(\mathbf{K})d}d\mathbf{K}. \tag{32}$$

This is the main equation for the Michelson. It will be illustrated below by means of examples obtained by choosing specific mathematical expressions for $S_{10}(m,\mathbf{K})$.

Problems involving reflecting or scattering objects other than an effectively infinite reflecting surface are important not only in microwave interferometry but also in other fields. Consequently the following formulation of the general case where the plane reflecting surface considered above is replaced by an arbitrary reflecting object is of interest. Let the general reflecting object be characterized by means of a scattering function $R(m,\mathbf{K};\ n,\ \mathbf{L})$ defined with $z=0$ as terminal surface. Instead of (27) we now have the linear transformation

$$a_m(\mathbf{K})=\int\sum R(m,\mathbf{K};n,\mathbf{L})\,b_n(\mathbf{L})\,d\mathbf{L} \tag{33}$$

as boundary condition. In operator notation (33) is written

$$\hat{a}=\hat{R}\hat{b}$$

and the equations corresponding to (28) are

$$b_0=S_{00}a_0+\hat{S}_{01}\hat{R}\hat{b}, \tag{34a}$$

$$\hat{b}=\hat{S}_{10}a_0+\hat{S}_{11}\hat{R}\hat{b}. \tag{34b}$$

The Liouville-Neumann series solution of (34b) may be written

$$\hat{b}=\hat{S}_{10}\hat{a}_0+\hat{L}\hat{S}_{10}\hat{a}_0, \tag{35}$$

where

$$\hat{L}=\sum_{p=1}^{\infty}(\hat{S}_{11}\hat{R})^p \tag{36}$$

is the operator corresponding to what is called the "resolvent kernel" in the theory of integral equations. Finally, for b_0 we obtain

$$b_0=S_{00}a_0+\hat{S}_{01}\hat{R}\hat{S}_{10}a_0+\hat{S}_{01}\hat{R}\hat{L}\hat{S}_{10}a_0. \tag{37}$$

In this form the last term (specifically the operator $\hat{L}$) accounts for multiple reflections.

An instance of (37) is the case of a large but finite mirror in the Michelson interferometer, which has been considered using an approximate expression for $\hat{R}$ and neglecting multiple reflections [8].

B. *Transmission systems; Fabry-Perot interferometer.* We now consider a general transmission system consisting of a radiating system and a receiving system with an arbitrary (electromagnetically linear) intervening structure or medium. For the active terminal the pertinent descriptive equation is (19b), which for convenience is repeated here:

$$\hat{b}=\hat{S}_{10}a_0+\hat{S}_{11}\hat{a}. \tag{19b}$$

This equation is understood to be set up with reference to a coordinate system Oxyz and space-side reference plane $z=0$, as detailed in section 2. Using primes to distinguish quantities associated with the passive terminal, we may write

$$b_0'=\hat{S}_{01}'\hat{a}', \tag{39a}$$

$$\hat{b}'=\hat{S}_{11}'\hat{a}', \tag{39b}$$

as the equations corresponding to (19). For these equations the space-side reference plane is $z=d$ in the above-mentioned coordinate system and the general arrangement is shown in figure 3. Next, let the structure and/or medium between the terminals be described by a set of linear operators $\hat{T}_{ij}$, defined with respect to $z=0$ and $z=d$ as reference planes, such that

$$\begin{aligned}\hat{a}&=\hat{T}_{11}\hat{b}+\hat{T}_{12}\hat{b}',\\ \hat{a}'&=\hat{T}_{21}\hat{b}+\hat{T}_{22}\hat{b}'.\end{aligned} \tag{40}$$

(The Fabry-Perot interferometer considered below will furnish an example of these equations. It will be a very special example, however, since specular reflection and transmission will be assumed, so that the operators $\hat{T}_{ij}$ will be diagonal and (40) is then reducible to a family of ordinary equations.)

A method of solving the problem described in (19), (39), and (40) may be indicated as follows. The result of eliminating $\hat{b}$ and $\hat{b}'$ from (19b), (39b), and (40) may be put in the form

$$\begin{bmatrix}\hat{a}\\ \hat{a}'\end{bmatrix}=\begin{bmatrix}\hat{T}_{11}\hat{S}_{10}a_0\\ \hat{T}_{21}\hat{S}_{10}a_0\end{bmatrix}+\begin{bmatrix}\hat{T}_{11}\hat{S}_{11} & \hat{T}_{12}\hat{S}_{11}'\\ \hat{T}_{21}\hat{S}_{11} & \hat{T}_{22}\hat{S}_{11}'\end{bmatrix}\begin{bmatrix}\hat{a}\\ \hat{a}'\end{bmatrix}. \tag{41}$$

As this form suggests, these equations may be solved for $\hat{a}$ and $\hat{a}'$ by a process of successive approximations similar to that used above. The received wave amplitude b_0' is directly determinable from (39a) as soon as $\hat{a}'$ is known. (b_0 is also directly determinable.)

To pass to the consideration of a highly simplified model of the Fabry-Perot interferometer, we first assume that $\hat{S}_{11}$ and $\hat{S}_{11}'$ are effectively so small that reflections at the terminals of the system may be neglected; that is, we assume that the second term on the right in (41) may be neglected. We then have the explicit expression

$$\hat{a}'=\hat{T}_{21}\hat{S}_{10}a_0, \tag{42}$$

and consequently for the received wave-amplitude we have

$$b_0'=\hat{S}_{01}'\hat{T}_{21}\hat{S}_{10}a_0. \tag{43}$$

Next, we assume that the structure described by the $\hat{T}_{ij}$ is a Fabry-Perot "etalon", consisting of a pair of elements corresponding to the two plates of an optical Fabry-Perot interferometer. Such elements might be, for example, perforated metal sheets or stacked quartz plates [8]. We assume that each element is symmetric with respect to the z-direction and characterizable by means of a (specular) reflection coefficient $\rho(m,\mathbf{K})$ and a (specular) transmission coefficient $\tau(m,\mathbf{K})$ defined at the symmetry plane of the element as reference plane. It is not assumed that the elements are lossless; ρ and τ are subject merely to realizability conditions for passive elements. We let the two elements be located so that their reference planes coincide with $z=0$ and with $z=d$, respectively. It may be noted that so locating the elements implies no real loss of generality, since the planes $z=0$ and $z=d$ are arbitrarily located with respect to the physical arrangements with which they are respectively associated. From the symmetry of the etalon as a whole with respect to $z=d/2$ and the fact that the $\hat{T}_{ij}$ must be diagonal we have

$$T_{11}=T_{22}=\delta_{mn}\delta(\mathbf{K}-\mathbf{L})t_{11}(m,\mathbf{K}) \tag{44a}$$

$$T_{21}=T_{12}=\delta_{mn}\delta(\mathbf{K}-\mathbf{L})t_{21}(m,\mathbf{K}), \tag{44b}$$

where $\delta(\mathbf{K}-\mathbf{L})$ means $\delta(k_x-l_x)\ \delta(k_y-l_y)$ and

$$t_{11}=\rho\,\frac{1+(\tau^2-\rho^2)e^{2i\gamma d}}{1-\rho^2e^{2i\gamma d}}, \tag{45a}$$

$$t_{21}=\frac{\tau^2e^{i\gamma d}}{1-\rho^2e^{2i\gamma d}}, \tag{45b}$$

as may be found by ordinary methods. In these expressions γ, ρ, and τ in general depend upon $\mathbf{K}$. When evaluated for $\mathbf{K}=0$ and simplified somewhat as they may be for lossless elements, the expressions reduce to ones frequently used in discussions of the optical Fabry-Perot interferometer.

Using (44b) and writing out (43) we obtain

$$b_0'=a_0\int\sum S_{01}'(m,\mathbf{K})t_{21}(m,\mathbf{K})S_{10}(m,\mathbf{K})d\mathbf{K};$$

finally, inserting (45b) and defining $\Psi(d)=b_0'/a_0$,

$$\Psi(d)=\int\sum\frac{S_{01}'(m,\mathbf{K})\tau^2(m,\mathbf{K})e^{i\gamma(\mathbf{K})d}S_{10}(m,\mathbf{K})d\mathbf{K}}{1-\rho^2(m,\mathbf{K})e^{2i\gamma(\mathbf{K})d}}. \tag{46}$$

This is the "transmission integral" for a Fabry-Perot interferometer. An example of the analytical evaluation of (46) will be given below, assuming constant ρ and τ and choosing very simple expressions for S_{01} and S_{10}. (It may be observed that in regarding (46) as a function of d, it is implicit that the terminal apparatuses remain fixed relative to the reference planes with which they are respectively associated.)

The examples that follow have been chosen to illustrate various features of the theory and of the instruments considered. Inasmuch as the examples are rather hypothetical in nature and mainly illustrative, no thorough or rigorous discussions are attempted.

C. *Examples, Michelson case.*

C.1. *Dipole.* The following example seems well-suited to illustrate the theory, inasmuch as it involves both TE and TM field-components in a fairly complicated way, the integrals involved can be evaluated, and the form of the answer can be anticipated. In this example the radiated field is assumed to be identical to that of an elementary electric dipole of moment $\mathbf{p}$ located at O.

To find the angular spectrum of the electric field we may proceed as follows. The appropriate Hertz potential is [9]

$$\mathbf{\Pi}=(4\pi\epsilon)^{-1}\mathbf{p}e^{ikr}/r;$$

the representation of the spherical wave $\exp(ikr)/r$ in terms of plane waves is [10], for $z>0$,

$$e^{ikr}/r=-(2\pi i)^{-1}\int\gamma^{-1}e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{K};$$

where $\mathbf{k}=\mathbf{k}^+$ is understood. Since $\mathbf{E}=\nabla\times\nabla\times\mathbf{\Pi}$ we have

$$\mathbf{E}=C\int\mathbf{k}\times(\mathbf{k}\times\mathbf{p})\gamma^{-1}e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{K}, \tag{47}$$

where $C=(8\pi^2\epsilon i)^{-1}$. Hence the complete vectorial spectrum is $\mathbf{b}=C\mathbf{k}\times(\mathbf{k}\times\mathbf{p})\gamma^{-1}$. This result holds for all $\mathbf{K}$; immediate confirmation for $K^2<k^2$ may be obtained from (25) using the asymptotic form

$$-k^2(4\pi\epsilon)^{-1}\mathbf{r}\times(\mathbf{r}\times\mathbf{p})e^{ikr}/r^3$$

for $\mathbf{E}$ [9].

According to the definition of $S_{10}(m, \mathbf{K})$, we have in this example

$$S_{10}(m,\mathbf{K})=(C/a_0)\boldsymbol{\kappa}_m\cdot[\mathbf{k}\times(\mathbf{k}\times\mathbf{p})]\gamma^{-1} \tag{48}$$

for $m=1, 2$. These spectral components are to be substituted into the reflection integral (32) for the Michelson instrument. The coefficient of $\exp(2i\gamma d)$ in the integrand of (32), after some vector-algebraic labor, is found in the present instance to be expressible in the form

$$-\omega\epsilon(C/a_0)^2\mathbf{p}\cdot[\mathbf{k}\times(\mathbf{k}\times\mathbf{p}')]\gamma^{-1},$$

where $\mathbf{p}'$ is the negative mirror image of $\mathbf{p}$, with components $(-p_x,-p_y, p_z)$. Thus for (32) we have

$$\Phi(d)=-\omega\epsilon\eta_0^{-1}(C/a_0)^2\mathbf{p}\cdot\int\mathbf{k}\times(\mathbf{k}\times\mathbf{p}')\gamma^{-1}e^{2i\gamma d}d\mathbf{K}. \tag{49}$$

Comparing this expression with (47) it is seen that

$$\Phi(d)=-\frac{C\omega\epsilon}{\eta_0 a_0^2}\mathbf{p}\cdot\mathbf{E}'(0,0,2d), \tag{50}$$

where $\mathbf{E}'$ is the electric field of a dipole of moment $\mathbf{p}'$ located at O, and $\mathbf{E}'$ is evaluated at the image of O in the reflecting surface of the instrument. This result is indeed of a form that might be anticipated.

If for simplicity we take $\mathbf{p}$ to be transverse, (50) becomes more explicitly [9],

$$\Phi(d)=C_1[(2ikd)^{-3}-(2ikd)^{-2}+(2ikd)^{-1}]e^{2ikd},$$

where C_1 is independent of d. It is clear that the result in this example is not physically meaningful as $kd\to 0$, for the magnitude of Φ can not properly exceed unity. This defect is attributable to the neglect of scattering or re-radiation by the dipole; a plausible extension of the theory of this example taking scattering into account gives results qualitatively well-behaved for all values of kd.

C.2 *Two-dimensional Gaussian.*—To provide a reasonably simple analytical illustration of the behavior of a Michelson instrument operated in the Fresnel region, we consider a two-dimensional case with the pure TM "Gaussian" spectrum

$$b_1(k_x)=A\,e^{-a^2k_x^2/2},$$

$$b_2(k_x)=0. \tag{51}$$

The subscripts 1, 2 here refer to $\mathbf{e}_x$, $\mathbf{e}_y$, respectively, and A is an arbitrary amplitude. The corresponding distribution of $\mathbf{E}_t$ on the reference plane $z=0$ is also Gaussian,

$$E_x=\text{Const. } e^{-x^2/(2a^2)},$$

$$E_y=0,$$

as follows from (14_2) (see appendix, B). The parameter a is, in a well-known manner, a measure of the sharpness of the angular spectrum and a measure of the width of the distribution of $\mathbf{E}_t$.

In this example it will be interesting to determine the absolute magnitude of Φ explicitly. We need the relation between $|a_0|$ and $|A|$ and we obtain this from a consideration of energy balance under free-space radiation conditions. If the fraction h of the net input power at S_0 is radiated, we have

$$\pi\eta_0|a_0|^2(1-|S_{00}|^2)h=\pi\int_{-k}^{k}\sum\eta_m|b_m(k_x)|^2dk_x, \tag{52}$$

where the left- and the right-hand sides of this equation come from (4_2) and (16_2), respectively.

The appropriate two-dimensional form of (32) is

$$\Phi(d)=-\eta_0^{-1}\int\sum\eta_m(k_x)S_{10}(m,k_x)S_{10}(m,-k_x)e^{2i\gamma(k_x)d}dk_x, \tag{53}$$

where we still have $\eta_1=\omega\epsilon\gamma^{-1}$ and $\eta_2=\gamma(\omega\mu)^{-1}$. In the present case this becomes

$$\Phi(d)=-\frac{A^2\omega\epsilon}{\eta_0 a_0^2}\int\gamma^{-1}\exp\,(-a^2k_x^2+2i\gamma d)dk_x.$$

Letting ϕ denote the phase of A/a_0 and using (52) to eliminate $|A/a_0|$ from the last equation, one obtains

$$\Phi(d)=-e^{2i\phi}(1-|S_{00}|^2)h\frac{\int\gamma^{-1}\exp\,(-a^2k_x^2+2i\gamma d)dk_x}{\int_{-k}^{k}\gamma^{-1}\exp\,(-a^2k_x^2)dk_x}. \tag{54}$$

If $ka>>1$, this expression yields $\Phi(0)\cong-e^{2i\phi}(1-|S_{00}|^2)h$—a quantity that may approximate unity in magnitude.

Since we are interested in Fresnel-region behavior, an asymptotic expansion of the numerator of (54) in terms of inverse powers of a is appropriate. This asymptotic expansion may be found with the aid of Watson's lemma, as given by Jeffreys and Jeffreys [11]. One finds

$$\Phi(d)=C_1e^{2ikd}\left[1+\frac{1-2ikd}{4(ka)^2}+\frac{9-18ikd-12(kd)^2}{32(ka)^4}+\cdots\right], \tag{55}$$

where C_1 does not depend upon d. For this expression to yield a good approximation it is necessary that $ka>>1$ and that $ka^2>>d$ (these inequalities are equivalent to those given in section 1 in the description of Fresnel-region operation). To the second order in ka,

$$\arg\,\Phi(d)\cong 2kd[1-(2ka)^{-2}].$$

Thus the "diffraction correction" to the phase can be expressed in terms of a small increase in effective wavelength, which in this approximation and in this example is independent of d.

It is of some interest to evaluate this result for values of k and a that might be considered typical of optical cases, even though the formula does not apply, or at least does not apply directly, to any optical instrument. If one takes $\lambda=5000$ Å and $a=5$ cm, the fractional increase in effective wavelength given by the formula is approximately 6×10^{-13}.

C.3 *TE_{10} aperture-distribution.* A somewhat more realistic—and much more intractable—example of Fresnel region behavior is afforded by the assumption of a TE_{10}-mode distribution in a square aperture in a conducting screen. This example is suggested by an experimental arrangement in which a square horn-lens radiator is fed by a taper from rectangular waveguide supporting the TE_{10} mode [8]. If the aperture is bounded by $|x|=a/2$ and $|y|=a/2$ in the plane $z=0$, we may take

$$\mathbf{E}_t=\cos(\pi x/a)\mathbf{e}_y$$

in the aperture and $\mathbf{E}_t=0$ elsewhere in the plane. From (22) and (23) it follows that

$$S_{10}(1,\mathbf{K})=k_yf(\mathbf{K})/|K|,$$

$$S_{10}(2,\mathbf{K})=k_xf(\mathbf{K})/|K|,$$

where

$$f(\mathbf{K})=C\frac{\cos\,(k_xa/2)}{(\pi/a)^2-k_x^2}\cdot\frac{\sin\,(k_ya/2)}{k_y}.$$

(Here and subsequently unimportant constant factors are denoted C, C', etc., without explicit definition in each case.) Hence (32) becomes

$$\Phi(d)=C'\int\left(\frac{\omega\epsilon}{\gamma}k_y^2+\frac{\gamma}{\omega\mu}k_x^2\right)\frac{[f(\mathbf{K})]^2}{K^2}e^{2i\gamma d}d\mathbf{K}. \tag{56}$$

This integral has been evaluated numerically. The numerical analysis and the programming required for this difficult task were performed by Paul F. Wacker and William W. Longley, Jr. Some of the results are shown in table 1, wherein the quantity labeled Δd is calculated in accordance with the definition

$$\Delta d=(2k)^{-1}[\arg\Phi(d)-\arg\Phi(0)]-d.$$

TABLE 1

λ	a	d	Δd
cm	*cm*	*m*	*Microns*
0.6278	60	2	−56.96
.6278	60	10	−193.80
.6278	30	2	−155.93
.6278	30	10	−503.28
.1	60	2	−2.32
.1	60	10	−7.56

The fact that this quantity is negative corresponds to the general fact that the effective wavelength tends to be greater than the free-space wavelength, so that the observed phase increases with d more slowly than $2kd$.

D. *Fabry-Perot with line source.* In this example the radiated field of the radiating system is taken to be identical to that of a simple line current coinciding with the y-axis, and it is assumed that the receiving system is the same as the radiating system. These assumptions yield a two-dimensional, pure TE problem in which the free-space radiation pattern has no Fresnel region. Although only the most readily obtainable results are given here, the example already affords an interesting illustration of the behavior of a Fabry-Perot instrument in the presence of a continuous angular spectrum.

The assumed field being pure TE, we certainly have $S_{10}(1,k_x)=0$; and since $\mathbf{E}$ is asymptotically proportional to $r^{-1/2}e^{ikr}\mathbf{e}_y$, it follows from ($25_2$) that we may write $S_{10}(2,k_x)=\gamma^{-1}$, at least for $k_x^2<k^2$. We shall employ this expression for the whole range of k_x, choosing to justify this analytic continuation by the results to be obtained in a moment. Using the reciprocity relation (20_2) we obtain for the receiving system $S'_{01}(2,k_x)=\gamma(\omega\mu)^{-1}\eta_0^{-1}S'_{10}(2,-k_x)=(\omega\mu\eta_0)^{-1}$. Hence the transmission integral (46) becomes

$$\Psi(d)=C\int_{-\infty}^{\infty}\frac{\tau^2e^{i\gamma d}\gamma^{-1}}{1-\rho^2e^{2i\gamma d}}\,dk_x. \tag{58}$$

The integral diverges for $d=0$; this behavior is attributable to the implicit assumption of an enforced current in the radiating element (as in the dipole example above). If we assume τ and ρ constants independent of k_x, $d>0$, and $|\rho|<1$, then (58) may be written

$$\Psi(d)=C\tau^2\sum_{n=0}^{\infty}\rho^{2n}\int_{-\infty}^{\infty}\gamma^{-1}e^{i(2n+1)\gamma d}dk_x.$$

By introducing a new variable of integration α, such that $k\sin\alpha=-\gamma$, one may transform the integral to one of the standard representations of the Hankel function [12] of the first kind and order zero. Thus

$$\Psi(d)=C'\tau^2\sum_{n=0}^{\infty}\rho^{2n}H_0^{(1)}[(2n+1)kd]. \tag{59}$$

At this point we note that for $\rho=0$, $\Psi(d)$ becomes proportional to $H_0^{(1)}(kd)$, as might be expected for free-space transmission between systems of the description considered.

For $|\rho|$ approaching unity, features peculiar to the Fabry-Perot may be expected to appear—and the series becomes very slowly convergent. A thorough discussion of (59) might be premature and will not be attempted here (one might wish to consider a more realistic model of the Fabry-Perot, for example). We shall assume $kd>>1$ and approximate the Hankel function by means of the first term of its asymptotic expansion. Instead of (59) we then have

$$\Psi(d)\cong C''\tau^2\rho^{-1}(kd)^{-\frac{1}{2}}F(\rho,kd), \tag{60a}$$

where

$$F(\rho,kd)=\sum_{n=0}^{\infty}(2n+1)^{-\frac{1}{2}}(\rho e^{ikd})^{2n+1}. \tag{60b}$$

In (60b) a phase angle in ρ obviously is equivalent to an additive constant in d; having noted this, we assume the ρ^2 is real and positive.

For $\rho^2<1$, the series ($60b$) converges for all values of d and $|F(\rho,kd)|$ has maxima for $kd=m\pi$, where m is an integer (for $\rho^2=1$, the series still converges for $kd\neq m\pi$ but diverges infinitely at the points where $kd=m\pi$). These maxima are certainly the principal maxima and correspond to the passage of axial rays; it is not known whether there are subsidiary maxima.

Since $|\tau|^2\leq 1-|\rho|^2$ (the equality holding for a lossless etalon), τ and hence Ψ must approach zero as $\rho^2\to 1$ except possibly at the values for d for which the series diverges. An estimate of $F(\rho,m\pi)$ as $\rho^2\to 1$ indicates that Ψ must approach zero at these points also. This decrease of transmission at the maxima as $\rho^2\to 1$ differs from the result given by the optical formula (45b) and may be surprising. It can be explained in terms of increasing selectivity for axial rays, such that a decreasing portion of the incident spectrum, and hence decreasing power, is transmitted.

Finally it may be observed that $F(\rho,kd)$ is a periodic function of d, so that the factor $d^{-\frac{1}{2}}$ in (60a) gives the general trend of Ψ with d. In the optical case, Ψ itself would be periodic.

4. Appendix

A. *Reciprocity relations.* Let $\mathbf{E}'$, $\mathbf{H}'$, and $\mathbf{E}''$, $\mathbf{H}''$ denote any two electromagnetic fields (of the same frequency) that can exist in the interior of the waveguide-space transducer considered. In virtue of the hypotheses imposed in section 2, the Lorentz relation

$$\nabla\cdot(\mathbf{E}'\times\mathbf{H}''-\mathbf{E}''\times\mathbf{H}')=0$$

will hold in V[6]. Therefore, using the divergence theorem, one has

$$\int_{S_0+S_0'+S_1+S_\infty}(\mathbf{E}'\times\mathbf{H}''-\mathbf{E}''\times\mathbf{H}')\cdot\mathbf{n}\,dS=0,$$

where $\mathbf{n}$ is the inward unit normal on the boundary of V and the parts S_0, S_0', etc., making up the boundary of V are as defined in section 2. Now, the contribution of the integral over S_0' vanishes, since the integrand vanishes there. By using the asymptotic form of the fields for large r, it can be shown that the S_∞-integral vanishes for fields generated by any distribution of sources confined to a sphere of finite radius centered at O. This restriction

apparently would rule out excitation of the system by plane waves incident on S_1, which we wish to consider, but since we can imagine a plane wave approximated arbitrarily closely by a source of finite dimensions at a sufficiently large distance, there is no real limitation. Hence we may employ the relation

$$\int_{S_0+S_1}(\mathbf{E}'\times\mathbf{H}''-\mathbf{E}''\times\mathbf{H}')\cdot\mathbf{n}\,dS=0 \qquad (61)$$

without explicit restrictions. If in this expression one replaces $\mathbf{E}'$, $\mathbf{H}'$ and $\mathbf{E}''$, $\mathbf{H}''$ by their representations on S_0 and S_1, equations (1) and (12), one finds after some analysis

$$\eta_0(a_0'b_0''-a_0''b_0')-\int\Sigma\eta_m(\mathbf{K})[a_m'(\mathbf{K})b_m''(-\mathbf{K}) -a_m''(\mathbf{K})b_m'(-\mathbf{K})]d\mathbf{K}=0. \qquad (62)$$

In obtaining this result the use of the integral representation (17) of the δ-function is helpful.

We now assume that $\mathbf{E}'$, $\mathbf{H}'$ and $\mathbf{E}''$, $\mathbf{H}''$ are the fields corresponding to excitation by the following particular sets of incident waves

$$a_0'=1, \qquad a_0''=0,$$

$$a_m'(\mathbf{K})=0; \qquad a_m''(\mathbf{K})=\delta_{mn}\delta(\mathbf{K}-\mathbf{L}).$$

From the scattering equations (18) we obtain

$$b_m'(\mathbf{K})=S_{10}(m,\mathbf{K}), \quad b_0''=S_{01}(n,\mathbf{L}).$$

Upon substituting all these quantities into (62) and observing that $\eta_m(\mathbf{L})=\eta_m(-\mathbf{L})$, one obtains the reciprocity relation (20) of the text.

In a very similar manner one can derive (21) of the text.

B. *Two-dimensional formulation.* For the discussion of two-dimensional cases many, if not most, of the formulas of the text require modification, and almost all the modifications may readily be obtained by reduction from formulas given in the text or by parallel development. Certain key formulas and minor subtleties are discussed here.

It is assumed that all quantities are independent of the transverse coordinates y and k_y, so that these coordinates will be absent from all formulas.

Under the above assumption the unit vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ designating the "parallel" and "perpendicular" electric-field directions degenerate to $\boldsymbol{\kappa}_1=\mathbf{e}_x$ sgn k_x, $\boldsymbol{\kappa}_2=\mathbf{e}_y$ sgn k_x. The inconvenience of the sign reversal is avoided by adopting $\mathbf{e}_x$, $\mathbf{e}_y$ as the basis vectors for the two polarizations. Sums over the polarization index m become sums over x- and y-components. This change induces a few further sign changes, the key one being in the reciprocity relation (given below).

The two-dimensional forms of (14) and (15) may be written

$$\mathbf{E}(\mathbf{r})_t=\int\mathbf{B}(k_x)e^{ik_xx+i\gamma z}dk_x, \qquad (14_2)$$

$$\mathbf{B}(k_x)=(2\pi)^{-1}e^{-i\gamma z}\int\mathbf{E}(\mathbf{r})_te^{-ik_xx}dx \qquad (15_2)$$

where $\mathbf{r}=x\mathbf{e}_x+z\mathbf{e}_z$, $\gamma^2=k^2-k_x^2$, and integrations with respect to k_x and x are understood to be taken over the range $(-\infty, \infty)$. (The numbering of these two equations indicates the scheme to be followed in this appendix.)

Power expressions such as (16) and (4) must be reinterpreted as power per unit length in the y-direction. Equation (16) becomes

$$P_r=\pi\int_{k_x^2<k^2}\Sigma\eta_m|b_m(k_x)|^2dk_x. \qquad (16_2)$$

It is convenient to match the loss of a factor of 2π between (16) and (16_2) by renormalizing the basis fields at S_0 so that (4) becomes

$$P_0=\pi\eta_0(|a_0|^2-|b_0|^2). \qquad (4_2)$$

With this renormalization the numerical factors in the reciprocity relation remain unchanged.

The reciprocity relation becomes

$$\eta_0S_{01}(m,k_x)=\eta_mS_{10}(m,-k_x). \qquad (20_2)$$

This is probably best established by a derivation parallel to that used for (20). The disappearance of a minus sign between the members of (20_2) as compared with (20) is due to the adoption of $\mathbf{e}_x$ and $\mathbf{e}_y$ as basis vectors for the two polarizations.

In the two-dimensional case the asymptotic form of $\mathbf{E}(\mathbf{r})_t$ for large r is

$$\mathbf{E}(\mathbf{r})_{t,\ \text{asymp}}=\left(\frac{2\pi}{ik}\right)^{\frac{1}{2}}k\cos\theta'r^{-\frac{1}{2}}e^{ikr}\mathbf{B}(k\sin\theta'), \qquad (24_2)$$

where $\theta'=\tan^{-1}(x/z)$. From this follows

$$\mathbf{S}_{10}(k_x)=\left(\frac{ik}{2\pi}\right)^{\frac{1}{2}}a_0^{-1}\gamma^{-1}r^{\frac{1}{2}}e^{-ikr}\mathbf{E}(\mathbf{k}r/k)_{t,\,\text{asymp}} \qquad (25_2)$$

for $k_x^2<k^2$, where $\mathbf{k}=k_x\mathbf{e}_x+\gamma\mathbf{e}_z$.

The power radiation pattern becomes

$$p=\pi(\epsilon/\mu)^{\frac{1}{2}}k\cos^2\theta'\mathbf{b}\cdot\bar{\mathbf{b}} \qquad (26_2)$$

where $\mathbf{b}$, the complete vectorial spectrum in the two-dimensional case, is $\mathbf{b}=b_x\mathbf{e}_x+b_z\mathbf{e}_z$ and $b_z=-k_xb_x\gamma^{-1}$.

The two-dimensional forms of equations such as (18), (32), and (46) may be written without difficulty. [The equation corresponding to (32) is written as (53) in the text.]

5. References and Notes

[1] For a general description of this work, see W. Culshaw, J. M. Richardson, and D. M. Kerns, Precision millimeter-wave interferometry at the U.S. National Bureau of Standards, Proc. of the Symposium on Interferometry at the National Physical Laboratory, Teddington, Middlesex, England (June 1959).

[2] A preliminary solution to the problem posed here using scalar waves and an asymptotic expansion of the reflection integral was obtained by Dayhoff (unpublished National Bureau of Standards Office of Basic Instrumentation report. This office supported the earlier phases of this work.)

[3] Microwave versions of the partially reflecting plates of the optical Fabry-Perot are discussed in the paper cited in note 1; see also W. Culshaw, Trans IRE, MTT-**7**, 221 (1959).

[4] See, e.g., J. A. Stratton, Electromagnetic theory, p. 361ff (McGraw-Hill Book Co., New York, N.Y., 1941).

[5] See e.g., B. Friedman, Principles and techniques of applied mathematics (John Wiley & Sons, New York, N.Y., 1956).

[6] D. M. Kerns, Basis of the application of network equations to waveguide problems, J. Research NBS, **42**, 515 (1949).

[7] This can be derived formally from (14) by application of methods originally devised for single integrals. For these methods see e.g., H. and B. S. Jeffreys, Methods of mathematical physics (Cambridge Univ. Press, 1950). For discussion of multiple integrals see D. S. Jones and M. Kline, Asymptotic expansion of multiple integrals and the method of stationary phase, N.Y.U., Inst. Math. Sci., Research Report No. EM–100 (1956).

[8] Discussed briefly in the reference cited in note 1.

[9] A convenient reference is J. A. Stratton, *op. cit.*, Chap. 8.

[10] This is readily derivable from, e.g., eq (26), p. 578, in Stratton, *op. cit.*

[11] H. and B. S. Jeffreys, *op. cit.*, p. 501ff.

[12] R. Courant and D. Hilbert, Methods of mathematical physics, 1st English Ed., p. 468 (Interscience Publishers, Inc., New York, 1953).

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Some Solutions for Electromagnetic Problems Involving Spheroidal, Spherical, and Cylindrical Bodies

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(September 29, 1959)

Solutions are presented for the low-frequency electromagnetic response to an oscillating magnetic dipole by conducting bodies of simple shape. The quasi-stationary approximation is employed throughout, which is valid when the relevant dimensions of the problem are all small compared to the free-space wavelength. This amounts to matching solutions of the wave equation within the bodies to solutions of Laplace's equation outside. The results have application to geophysical prospecting.

1. Introduction

Electromagnetic methods of geophysical exploration utilize the fact that the conductivity of massive ore bodies is much greater than the surrounding barren rock. The general scheme is to set up a primary or exciting field by a current-carrying loop and then to detect the secondary field or response of the body by means of a receiving loop. The operating frequency should be sufficiently low that the attenuation by the surrounding barren rock is negligible. This usually requires frequencies in the audio range. The literature on the subject is extensive and here only certain representative papers are referenced [1–4].[1]

Theoretical approaches to the subject are usually restricted to highly simplified situations. For example the exciting field is often assumed to be uniform or the body is taken to be perfectly conducting. While such solutions do indeed provide much useful information, there is a need to consider situations of a more general nature. For example, in the practical methods of electromagnetic prospecting the transmitting and receiving coils or loops may be located at distances from an ore body which are comparable to its maximum dimension and thus the uniform-exciting-field assumption is not valid. Furthermore, the frequency which is of the order of 500 cps is sufficiently low that the ore bodies seldom behave as if they were of perfect conductivity. For the above reasons, it seems worthwhile to set up solutions for certain idealized cases which do not suffer from such over-simplifying assumptions which are usually present. The geometrical forms considered are the prolate spheroid, the sphere, and the circular cylinder. Because of complexity the spheroid is taken to be perfectly conducting. The sphere and cylinder are assigned a conductivity and magnetic permeability which are finite. In each case the solution is presented for the case of an arbitrarily located magnetic dipole. The results are in a form which is suitable for computation.

In each case treated, special attention is paid to the *equatorial plane* which contains the source and observer. In the three instances, the source is at C and the observer at P as indicated in figure 1. For a (y-directed) magnetic dipole of strength $\overline{\mathrm{K}}$ at C, the primary field at P is

$$H_y^p = -\frac{\overline{K}dl}{2\pi s^3} \text{ (amp/m,)}$$

where $\overline{\mathrm{K}}dl$ is the magnetic moment. Note that $\overline{\mathrm{K}}dl$ = (amp-turns) × (coil area). In what follows the secondary fields are expressed in terms of x (distance from coil axes to center of body), y and y' (coordinates of source and observer), s (distance between C and P), σ conductivity of body (mhos/m), and μ/μ_0 (magnetic permeability ratio).

[1] Figures in brackets indicate the literature references at the end of this paper.

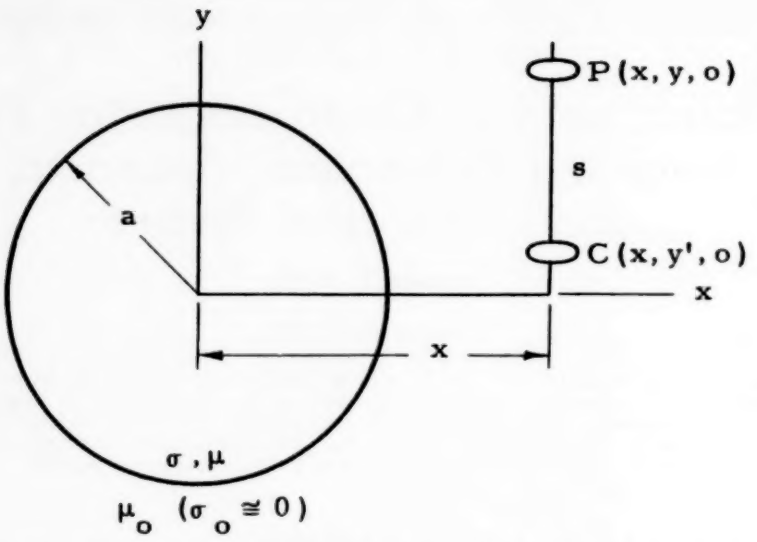

FIGURE 1.—*Cross section of spheroid, cylinder, or sphere in equatorial plane* ($z=0$), *showing transmitting and receiver coils.*

2. Low-Frequency Electromagnetic Response of a Highly-Conducting Spheroid

Prolate spheroidal coordinates are introduced (ρ, η, δ). They are defined by

$$\begin{aligned} &\frac{\rho^2}{c^2(\eta^2-1)}+\frac{z^2}{c^2\eta^2}=1 \qquad \text{(eq of spheroids)},\\ &\frac{\rho^2}{-c^2(1-\delta^2)}+\frac{z^2}{c^2\delta^2}=1 \qquad \text{(eq of hyperboloids)}, \end{aligned} \tag{1}$$

where ρ and z are the usual radial and axial coordinates in a cylindrical coordinate system. It thus follows that

$$\rho=c[(1-\delta^2)(\eta^2-1)]^{\frac{1}{2}} \qquad \text{and} \qquad z=c\eta\delta. \tag{2}$$

where c is the semifocal distance. These prolate spheroidal coordinates are taken to be confocal with the spheroidal body whose surface is defined by $\eta=\eta_0$ (see fig. 2).

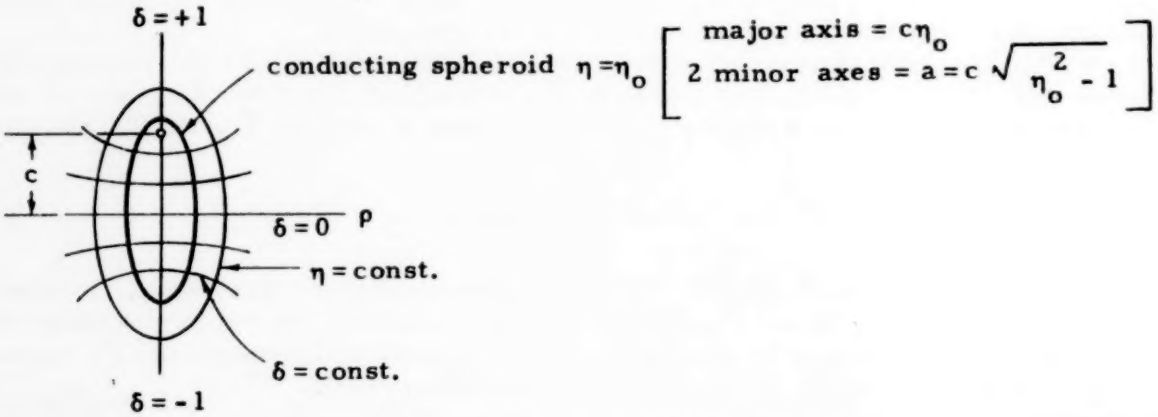

FIGURE 2.—*The spheroid and spheroidal coordinate system.*

In the region exterior to the body (i.e., $\eta > \eta_0$) the fields are a solution of Laplace's equation if the frequency is sufficiently low (i.e., significant distances should be much less than the wavelength). Furthermore, in this case the fields can be derived from a magnetic potential Ω. Thus,

$$\vec{H} = -\text{grad } \Omega$$

where

$$\nabla^2 \Omega = 0 \qquad \text{for} \qquad \eta > \eta_0.$$

Now, solutions of Laplace's equation in spheroidal coordinates are well known and are of the form (for integral values of m and n)

$$[AP_n^m(\delta) + BQ_n^m(\delta)][A'P_n^m(\eta) + B'Q_n^m(\eta)]\, e^{\pm im\phi}, \tag{3}$$

where A, B, A', and B' are constants and, following Smythe [5], $P_n^m(\mu)$ and $Q_n^m(\mu)$ are associated Legendre functions of argument μ. They are defined by

$$\left.\begin{aligned} P_n^m(\mu) &= (\mu^2-1)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m} \\ Q_n^m(\mu) &= (\mu^2-1)^{m/2} \frac{d^m Q_n(\mu)}{d\mu^m} \end{aligned}\right\} \text{for } \mu > 1 \tag{4}$$

or

$$\left.\begin{aligned} P_n^m(\mu) &= (1-\mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m} \\ Q_n^m(\mu) &= (1-\mu^2)^{m/2} \frac{d^m Q_n(\mu)}{d\mu^m} \end{aligned}\right\} \text{for} -1 < \mu < 1 \tag{5}$$

in terms of the (ordinary) Legendre functions $P_n(\mu)$ and $Q_n(\mu)$. It should be noted Hobson introduces a factor $(-1)^m$ on the right-hand side of the latter two equations and his definitions are thus slightly different.

At the point C it is assumed that there is a magnetic point charge of strength, $\overline{K}$ (see fig. 3). The magnetic potential at P can now be written

$$\Omega = \frac{\overline{K}}{4\pi R} + \Omega', \tag{6}$$

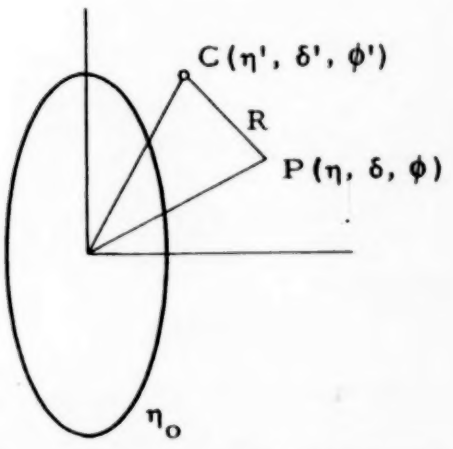

FIGURE 3.—*Conducting spheroid and source C and observer P.*

where R is the linear distance between C at (η', δ', ϕ') and P at (η, δ, ϕ) and Ω^s is the secondary influence of the body. To express Ω in prolate-spheroidal coordinates, we make use of the inverse-distance formula

$$\frac{1}{R}=\frac{1}{c}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(2-\delta_m)(-1)^m\left[\frac{(n-m)!}{(n+m)!}\right]^2(2n+1)P_n^m(\delta)Q_n^m(\eta')P_n^m(\eta)P_n^m(\delta')\cos m(\phi-\phi'), \quad (7)$$

which is valid for $\eta<\eta'$.[$\delta_0=1$, $\delta_m=0$ $(m\neq 0)$].

The potential Ω is a solution of Laplace's equation and since $1/R$ is also, it follows that $\nabla^2\Omega^s=0$ for $\eta>\eta_0$. Thus Ω^s must contain terms of the type

$$Q_n^m(\eta)P_n^m(\delta)\cos m(\phi-\phi')$$

only since $P_n^m(\eta)$ is infinite at $\eta\to\infty$ and $Q_n^m(\delta)$ is infinite at $\delta=\pm 1$. The potential exterior to the spheroid is thus written, for $\eta<\eta'$,

$$\Omega=\frac{\overline{K}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}[M_{mn}P_n^m(\eta)+B_{mn}Q_n^m(\eta)]P_n^m(\delta)\cos m(\phi-\phi'), \quad (8)$$

where

$$M_{mn}=\frac{1}{c}(2-\delta_m)(-1)^m(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2Q_n^m(\eta')P_n^m(\delta'). \quad (9)$$

The unknown coefficient is now found from the boundary condition, that on the surface of the spheroid the normal component of the total magnetic field must vanish. Thus,

$$\frac{\partial\Omega}{\partial\eta}=0 \qquad \text{at} \qquad \eta=\eta_0. \quad (9)$$

This is satisfied if

$$B_{mn}=-M_{mn}\frac{P_n^{m\prime}(\eta_0)}{Q_n^{m\prime}(\eta_0)}, \quad (10)$$

where the prime indicates the derivative with respect to the argument of the Legendre function. The solution is thus given by

$$\Omega=\frac{\overline{K}}{4\pi}\left[\frac{1}{R}-\sum_{n=0}^{\infty}\sum_{m=0}^{n}M_{mn}\frac{P_n^{m\prime}(\eta_0)}{Q_n^{m\prime}(\eta_0)}Q_n^m(\eta)P_n^m(\delta)\cos m(\phi-\phi')\right], \quad (11)$$

valid for $\eta>\eta_0$. To simplify things a bit we will consider our source to be a y-directed magnetic dipole T located in the equatorial plane of the spheroid $(z=0)$ (see fig. 4).

The potential at R (x, y, z) is thus given by

$$\begin{aligned}\Phi&=-dl\frac{\partial}{\partial y'}\Omega\\&=+\frac{\overline{K}dl}{4\pi}\left\{-\frac{\partial}{\partial y'}\frac{1}{R}+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{P_n^{m\prime}(\eta_0)}{Q_n^{m\prime}(\eta_0)}Q_n^m(\eta)\right.\\&\left.\times P_n^m(\delta)\left[\frac{\partial M_{mn}}{\partial y'}\cos m(\phi-'\phi')+\frac{\partial}{\partial y'}\cos m(\phi-\phi')\cdot M_{mn}\right]\right\}. \quad (12)\end{aligned}$$

The magnetic field at R is obtained from

$$\vec{H}=-\text{grad }\Phi.$$

In the case of the y-component, for example, we have

$$H_y=-\frac{\partial\Phi}{\partial y}=dl\frac{\partial^2\Omega}{\partial y'\partial y}. \quad (13)$$

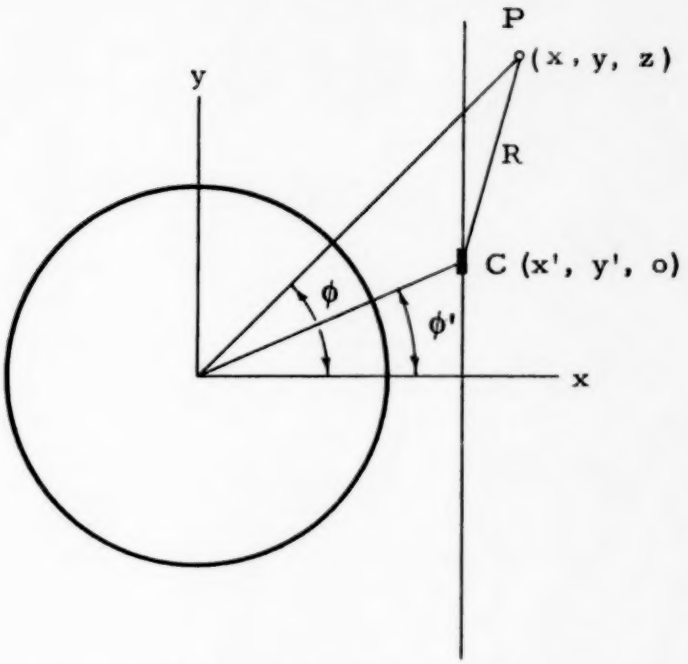

FIGURE 4.—*The conducting spheroid with locations of source C and observer P.*

But

$$\frac{\partial\Omega}{\partial y}=\frac{\partial\Omega}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}+\frac{\partial\Omega}{\partial\phi}\cdot\frac{\partial\phi}{\partial y}, \tag{14}$$

and therefore

$$\frac{\partial^2\Omega}{\partial y\partial y'}=\frac{\partial^2\Omega}{\partial\eta'\partial\eta}\left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\eta'}{\partial y'}\right)+\frac{\partial^2\Omega}{\partial\phi'\partial\phi}\left(\frac{\partial\phi}{\partial y}\right)\left(\frac{\partial\phi'}{\partial y'}\right)+\frac{\partial^2\Omega}{\partial\phi'\partial\eta}\left(\frac{\partial\eta}{\partial y}\right)\left(\frac{\partial\phi'}{\partial y'}\right)+\frac{\partial^2\Omega}{\partial\eta'\partial\phi}\left(\frac{\partial\phi}{\partial y}\right)\left(\frac{\partial\eta'}{\partial y'}\right). \tag{15}$$

Since

$$y=c\sqrt{(1-\delta^2)(\eta^2-1)}\,\sin\phi$$

and

$$x=c\sqrt{(1-\delta^2)(\eta^2-1)}\,\cos\phi,$$

it readily follows that for $\delta=0$ (or $z=0$),

$$\frac{\partial\eta}{\partial y}=\frac{\sin\phi}{c}\frac{\sqrt{\eta^2-1}}{\eta}\qquad\text{and}\qquad\frac{\partial\phi}{\partial y}=\frac{\cos\phi}{c}\frac{1}{\sqrt{\eta^2-1}} \tag{16}$$

and similarly for the primed quantities. Thus,

$$H_y=dl\,\frac{\partial^2\Omega}{\partial y'\partial y}$$

for a y-directed dipole in the plane $z=0$,

$$\Omega=\frac{\overline{K}}{4\pi R}+\Omega^s,$$

where

$$\Omega^r=-\frac{\overline{K}}{4\pi c}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m(-1)^m(2n+1)\left[\frac{(r-m!)}{(n+m)!}\right]^2\frac{P_n^{m\prime}(\eta_0)}{Q_n^{m\prime}(\eta_0)}[P_n^m(0)]^2\cos m(\phi-\phi')Q_n^m(\eta')Q_n^m(\eta), \quad (17)$$

where

$$\epsilon_m=2-\delta_m=1 \quad \text{for} \quad m=0$$
$$=2 \quad \text{for} \quad m\neq 0$$

and

$$\delta=\delta'=0.$$

Now

$$\frac{\partial^2}{\partial y\partial y'}Q(\eta)Q(\eta')\cos m(\phi-\phi')=V_n^m(\eta,\eta',\phi-\phi')$$

$$=Q'(\eta)Q'(\eta')\cos m(\phi-\phi')\left(\frac{\sin\phi}{c}\frac{\sqrt{\eta^2-1}}{\eta}\right)\left(\frac{\sin\phi'}{c}\frac{\sqrt{(\eta')^2-1}}{\eta'}\right)$$

$$+Q(\eta)Q(\eta')m^2\cos m(\phi-\phi')\left(\frac{\cos\phi}{c}\frac{1}{\sqrt{\eta^2-1}}\right)\left(\frac{\cos\phi'}{c}\frac{1}{\sqrt{(\eta')^2-1}}\right)$$

$$-Q(\eta)Q'(\eta')m\sin m(\phi-\phi')\left(\frac{\cos\phi}{c}\frac{1}{\sqrt{\eta^2-1}}\right)\left(\frac{\sin\phi'}{c}\frac{\sqrt{(\eta')^2-1}}{\eta'}\right)$$

$$+Q'(\eta)Q(\eta')m\sin m(\phi-\phi')\left(\frac{\sin\phi}{c}\frac{1}{\sqrt{\eta^2-1}}\right)\left(\frac{\cos\phi'}{c}\frac{1}{\sqrt{(\eta')^2-1}}\right) \quad (18)$$

where $Q=Q_n^m$.

Finally,

$$H_y=H_y^p+H_y^s$$

where

$$H_y^p=\frac{\overline{K}dl}{4\pi}\frac{\partial^2}{\partial y\partial y'}\frac{1}{R}=-\frac{\overline{K}dl}{4\pi}\left[\frac{3(y-y')^2}{R^5}-\frac{1}{R^3}\right]$$

$$=-\frac{\overline{K}dl}{2\pi s^3} \quad \text{for} \quad x'=x. \quad (19)$$

Noting that

$$R=|y'-y|=s=y-y'$$

and

$$H_y^s=-\frac{\overline{K}}{4\pi c}\sum_{n-0}^{\infty}\sum_{m-0}^{n}\epsilon_m(-1)^m(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2\frac{P_n^{m\prime}(\eta_0)}{Q_n^{m\prime}(\eta_0)}[P_n^m(0)]^2V_n^m \quad (20)$$

where V_n^m is as given above.

For the special case (i.e., $z=z'=0, |y'-y|=s$, and $x=x'$)
as shown in figure 5, we have

$$a^2=c^2(\eta_0^2-1), \qquad a=c\sqrt{\eta_0^2-1},$$

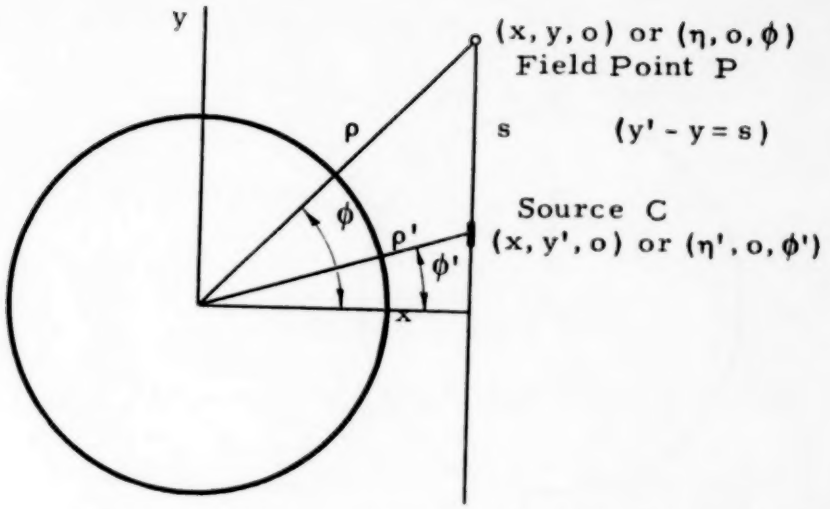

FIGURE 5.—*Special configuration of (dipole) source C and observer P when they are located in equatorial plane of spheroid.*

where a is the semiminor axis, c is the semifocal distance,

$$\rho^2=c^2(\eta^2-1), \qquad \rho=c\sqrt{\eta^2-1},$$

and

$$(\rho')^2=c^2[(\eta')^2-1], \qquad \rho'=c\sqrt{(\eta')^2-1}.$$

Also it should be noted that

$$s=[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')]^{\frac{1}{2}},$$

$$\tan\phi'=\frac{y'}{x} \quad \text{and} \quad \tan\phi=\frac{y}{x}.$$

If b is the semimajor axis,

$$b=c\eta_0.$$

3. Low-Frequency Electromagnetic Response of a Finitely-Conducting Sphere With Arbitrary Permeability

Spherical coordinates (r, θ, ϕ) are introduced as indicated in figure 6. The surface of a spherical body (of conductivity σ and permeability μ) is defined by $r=a$. Exterior to the sphere the fields can be derived from a magnetic potential Ω. Thus,

where
$$\left.\begin{aligned} H&=-\text{grad}\,\Omega \\ \nabla^2\Omega&=0 \end{aligned}\right\} \text{ for } r>a.$$

Inside the sphere the fields are a solution of the wave equation and are best derived in terms of scalar stream functions (following Schelkunoff) [6]. In the present problem we need only

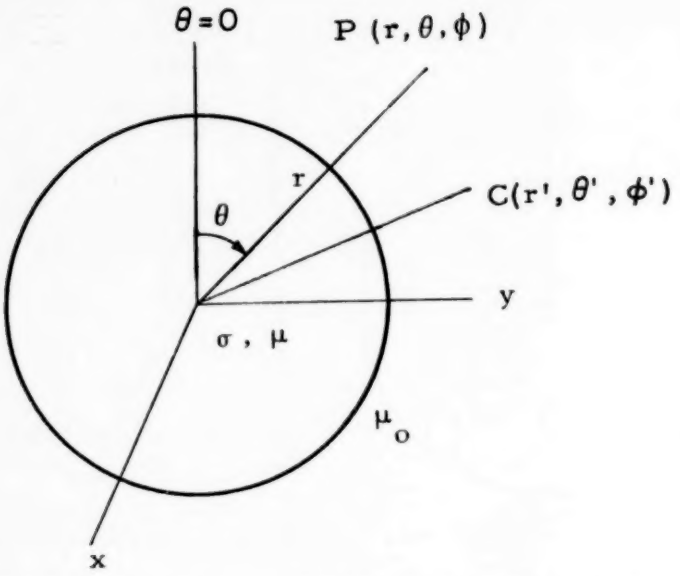

FIGURE 6.—*The spherical body and source C and observer P with the spherical coordinate system.*

consider the TE modes which are associated with the induced magnetic dipoles. Thus,

$$\left.\begin{aligned} H_r &= \frac{1}{i\mu\omega}\left(\frac{\partial^2}{\partial r^2}-\gamma^2\right)\psi \\ &= \frac{1}{i\mu\omega r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi \\ H_\theta &= \frac{1}{i\mu\omega}\frac{\partial^2\psi}{\partial r\,\partial\theta} \\ H_\phi &= \frac{1}{i\mu\omega r\sin\theta}\frac{\partial^2\psi}{\partial r\,\partial\phi} \end{aligned}\right\} \text{for } r<a, \tag{21}$$

where ψ is the stream function and

$$\gamma=[i\mu\omega(\sigma+i\omega\epsilon)]^{\frac{1}{2}}\simeq(i\sigma\mu\omega)^{\frac{1}{2}}, \tag{22}$$

whereas

$$\left.\begin{aligned} H_r &= \frac{1}{i\mu_0\omega}\frac{\partial^2}{\partial r^2}\psi_0=-\frac{\partial\Omega}{\partial r} \\ &= \frac{1}{i\mu_0\omega r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi_0, \\ H_\theta &= \frac{1}{i\mu_0\omega r}\frac{\partial^2\psi_0}{\partial r\partial\theta}=-\frac{\partial\Omega}{r\partial\theta}, \\ H_\phi &= \frac{1}{i\mu_0\omega r\sin\theta}\frac{\partial^2\psi_0}{\partial r\partial\phi}=-\frac{\partial\Omega}{r\sin\theta\partial\phi}, \end{aligned}\right\} \text{for } r>a, \tag{23}$$

where ψ_0 is the stream function pertinent to the exterior region and is related to the magnetic potential by

$$\Omega=-\frac{1}{i\mu_0\omega}\frac{\partial\psi_0}{\partial r}. \tag{24}$$

The stream functions satisfy

$$\left[r^2\frac{\partial^2}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi=\gamma^2r^2\psi \qquad \text{for } r<a$$
$$\psi_0=0 \qquad \text{for } r>a. \tag{25}$$

Thus they are of the form

$$\begin{matrix}\hat{I}_n(\gamma r)\\ \hat{K}_n(\gamma r)\end{matrix} P_n^m(\cos\theta) \qquad \text{for} \qquad r<a \tag{26}$$

and

$$\begin{matrix}r^{-n}\\ r^{+(n+1)}\end{matrix} P_n^m(\cos\theta) \qquad \text{for} \qquad r>a. \tag{27}$$

In the above $\hat{I}_n$ and $\hat{K}_n$ are the spherical modified Bessel functions defined by

$$\hat{I}_n(z)=\frac{1}{2}\left[e^z\sum_{m=0}^{n}\frac{(-1)^m(n+m)!}{m!(n-m)!(2z)^m}+(-1)^{n+1}e^{-z}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!(2z)^m}\right] \tag{28}$$

and

$$\hat{K}_n(z)=e^{-z}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!(2z)^m}, \tag{29}$$

following Schelkunoff [6].

Now for $r>a$,

$$\Omega=\Omega^p+\Omega^s, \tag{30}$$

where the primary source is denoted Ω^p and the influence of the sphere by Ω^s. Assuming a magnetic charge (strength $\overline{K}$) located at point C with coordinates (r', θ', ϕ'), the potential Ω^p at point P with coordinates (r, θ, ϕ) is given by

$$\Omega^p=\frac{\overline{K}}{4\pi R}$$
$$=\frac{\overline{K}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\frac{r^n}{(r')^{n+1}}P_n^m(\cos\theta)P_n^m(\cos\theta')\times\cos m(\phi-\phi') \tag{31}$$

for $r<r'$, using a well-known addition theorem for the inverse distance in spherical harmonics [5]. This suggests writing

$$\Omega^s=\frac{\overline{K}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\left(\frac{n}{n+1}\right)S_n\frac{a^{2n+1}}{(r')^{n+1}r^{n+1}}\times P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi'), \tag{32}$$

where S_n is an undetermined coefficient. The appropriate form for ψ_0 is thus

$$\psi_0=-\frac{\overline{K}i\mu_0\omega}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\left\{\left(\frac{r}{r'}\right)^{n+1}-\frac{a^{2n+1}}{(r')^{n+1}r^n}S_n\right\}\frac{P_n^m(\cos\theta)P_n^m(\cos\theta')}{(n+1)}\times\cos m(\phi-\phi'). \tag{33}$$

For the interior the solution must be of the form

$$\psi=-\frac{\overline{K}i\mu_0\omega}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!\hat{I}_n(\gamma r)}{(n+m)!\hat{I}_n(\gamma a)}\frac{P_n^m(\cos\phi)P_n^m(\cos\phi')}{(n+1)}B_n\left(\frac{a}{r'}\right)^{n+1}, \tag{34}$$

where the $\hat{K}_n(\gamma r)$ solution has been rejected since it becomes infinite at $r=0$ and where B_n is an undetermined coefficient.

The boundary conditions require that the normal flux (μH_r) inside the sphere is continuous with the exterior normal flux ($\mu_0 H_r$) at $r=a$. Also the tangential fields H_θ and H_ϕ must be continuous at $r=a$. These conditions are satisfied if

$$\mu \frac{\partial \psi}{\partial r}\Big]_{r=a} = \mu_0 \frac{\partial \psi_0}{\partial r}\Big]_{r=a} \tag{35}$$

and

$$\psi]_{r=a} = \psi_0]_{r=a}. \tag{36}$$

This leads to

$$S_n = \frac{A_n(\alpha) - (n+1)(\mu/\mu_0)}{A_n(\alpha) + n(\mu/\mu_0)} \tag{37}$$

where

$$A_n(\alpha) = \frac{\alpha \frac{d\hat{I}_n(\alpha)}{d\alpha}}{\hat{I}_n(\alpha)} = \alpha \frac{d}{d\alpha}[\log_e \hat{I}_n(\alpha)] \tag{38}$$

with $\alpha = \gamma a$. Also,

$$B_n = \frac{(\mu/\mu_0)}{n(\mu/\mu_0) + A_n(\alpha)}. \tag{39}$$

The above expressions can now be used to obtain an expression for the magnetic potential due to a dipole for $r > a$ in terms of the potential Ω due to a point charge. For example in the case of a y directed dipole at (r', ϕ', z') we have

$$\Phi_0 = -dl \frac{\partial}{\partial y'} \Omega. \tag{40}$$

The corresponding magnetic fields are thus given by

$$\vec{H} = -\text{grad } \Phi_0 \tag{41}$$

and the y-component, for example, is

$$H_y = dl \frac{\partial \Omega}{\partial y \partial y'}. \tag{42}$$

Again we shall consider the equatorial plane (see fig. 7) which is $z = z' = 0$ or

$$\theta = \theta' = \pi/2. \quad \text{Here } x = r\cos\phi \qquad x' = r'\cos\phi'$$
$$y = r\sin\phi \qquad y' = r'\sin\phi',$$

and thus,

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial r} + \cos\phi \frac{\partial}{\rho \partial \phi} \tag{43}$$

and

$$\frac{\partial}{\partial y'} = \sin\phi' \frac{\partial}{\partial r'} + \cos\phi' \frac{\partial}{\rho' \partial \phi'}. \tag{44}$$

The field is thus obtained conveniently from

$$H_y = dl\left[\sin\phi' \sin\phi \frac{\partial^2}{\partial r' \partial r} + \sin\phi' \cos\phi \frac{\partial^2}{\rho \partial r' \partial \phi} + \cos\phi' \sin\phi \frac{\partial^2}{\rho' \partial \phi' \partial r} + \cos\phi' \cos\phi \frac{\partial^2}{\rho' \rho \partial \phi \partial \phi'}\right]\Omega. \tag{45}$$

It should be noted that

$$\rho'=r' \qquad \text{for} \qquad \theta'=\pi/2$$
$$\rho=r \qquad \text{for} \qquad \theta=\pi/2. \tag{46}$$

Writing

$$H_y=H_y^p+H_y^s,$$

it is seen that

$$H_y^p=\overline{K}\frac{dl}{4\pi}\frac{\partial^2}{\partial y\partial y'}\frac{1}{R} \qquad \text{where} \qquad R=\sqrt{(x-x')^2+(y-y')^2} \qquad \text{for} \qquad z=z'=0, \tag{47}$$

and thus,

$$H_y^p=\frac{\overline{K}dl}{4\pi}\frac{\partial}{\partial y'}\left(\frac{y-y'}{R^3}\right)=-dl\left[-\frac{1}{R^3}+\frac{3(y-y')^2}{R^5}\right]\frac{\overline{K}}{4\pi}$$
$$=-\frac{\overline{K}dl}{4\pi s^3} \qquad \text{where} \qquad s=|y-y'|. \tag{48}$$

The secondary field is derived from

$$\Omega_0^s=\frac{\overline{K}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\frac{a^{2n+1}n}{(r')^{n+1}r^{n+1}}S_n\frac{P_n^m(\cos\theta)P_n^m(\cos\theta')}{n+1}\cos m(\phi-\phi'). \tag{49}$$

Finally (for $r\geqq a$)

$$H_y^s=\frac{\overline{K}}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\epsilon_m\frac{(n-m)!}{(n+m)!}\frac{n}{n+1}S_n[P_n^m(0)]^2U_{nm}\frac{a^{2n+1}}{(r')^{n+1}r^{n+1}} \tag{50}$$

where

$$U_{nm}=\frac{1}{rr'}[(n+1)^2\sin\phi'\sin\phi\cos m(\phi-\phi')$$
$$+m(n+1)\sin\phi'\cos\phi\sin m(\phi-\phi')$$
$$-m(n+1)\cos\phi'\sin\phi\sin m(\phi-\phi')$$
$$+m^2\cos\phi'\cos\phi\cos m(\phi-\phi')], \tag{51}$$

where

$$\tan\phi=\frac{y}{x},$$

$$\tan\phi'=\frac{y'}{x}.$$

The quantity S_n plays rather an essential role in the final result. When the conductivity is great or at high frequency, the argument γa of the Bessel functions may be large. In fact if $|\gamma a|>>1$, S_n approaches unity for all n. The behavior of S_n, as a function of γa and μ/μ_0, is a convenient way to illustrate the frequency dependence of the secondary fields. Numerical values of the real and imaginary parts of S_n are given in some detail for $n=1$ in previous papers [2, 8] and less adequately [9] for $n=2$, 3, and 4. Fortunately, if the source C or the observer P (for any respective orientations) are a distance from the center of the sphere which is large compared to the radius, it is seen that only the term corresponding to S_1 is important. The response of the sphere is thus proportional to the quantity

$$S_1=\frac{\gamma a\hat{I}_1'(\gamma a)-2K\hat{I}_1(\gamma a)}{\gamma a\hat{I}_1'(\gamma a)+K\hat{I}_1(\gamma a)} \qquad \text{where} \qquad K=\mu/\mu_0.$$

The real and imaginary parts of S_1 are plotted in figure 8 as a function of the "relative radius" X defined by

$$\gamma a=\sqrt{i}X \qquad \text{or} \qquad X=(\sigma\mu\omega)^{\frac{1}{2}}a.$$

For small values of X corresponding to the lower frequencies, S_1 is negative and, in fact, it approaches the value $-2(K-1)/(K+2)$. This could have been derived directly from magnetostatics. At the higher frequencies S_1 approaches unity which corresponds to no penetration of magnetic flux into the body.

4. Low-Frequency Response of a Conducting Cylinder With Arbitrary Permeability

We now wish to calculate the electromagnetic response of an infinitely long conducting cylinder with any permeability. (See fig. 9.) Exterior to the cylinder $\rho > a$, the fields are a solution of Laplace's equation and are derivable from a potential. Inside the cylinder the fields are a solution of the wave equation and are derivable from a magnetic Hertz vector $\vec{\Pi}^*$ and an electric Hertz vector $\vec{\Pi}$ with only z-components. Thus (for $r < a$)

$$\vec{H} = (-\gamma^2 + \text{grad div})\vec{\Pi}^* + \frac{\gamma^2}{i\mu\omega} \text{ curl } \vec{\Pi}, \tag{52}$$

where

$$\vec{\Pi}^* = (0,0,\Pi_z^*) \tag{53}$$

and

$$\vec{\Pi} = (0,0,\Pi_z). \tag{54}$$

Solutions of the wave equation

$$(\nabla^2 - \gamma^2)\begin{matrix}\Pi_z^* \\ \Pi_z\end{matrix} = 0 \tag{55}$$

are of the form

$$\begin{matrix}I_m(\alpha\rho) \\ K_m(\alpha\rho)\end{matrix} e^{-im\phi} e^{-ihz} \tag{56}$$

where $\alpha = (\gamma^2 + h^2)^{\frac{1}{2}}$ and I_m and K_m are cylindrical Bessel functions of the modified kind defined by G. N. Watson [7].

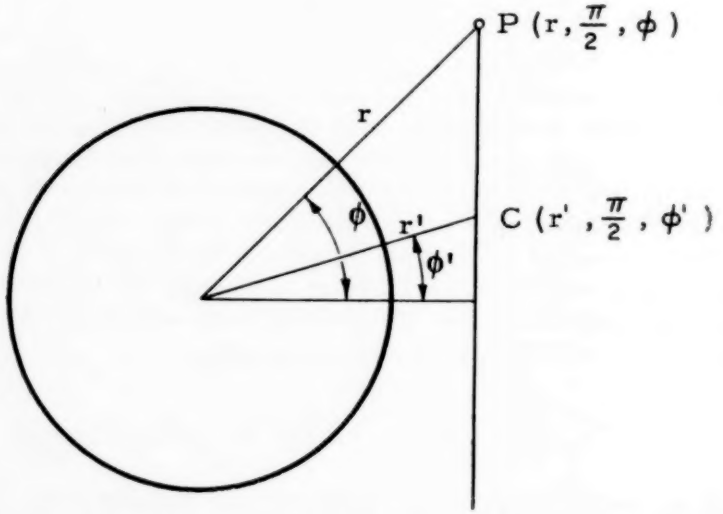

FIGURE 7.—*Special configuration of (dipole) source C and observer P when they are located in equatorial plane of sphere.*

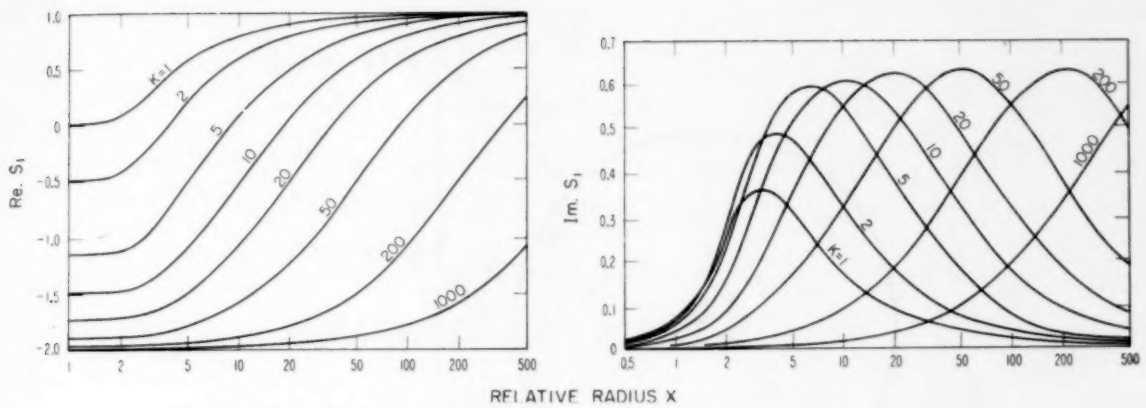

FIGURE 8.—*The variation of the response of a conducting sphere as a function of* $X(=(\sigma\mu\omega)^{\frac{1}{2}}a)$ *for various permeability ratios* $K(=\mu/\mu_0)$.

The real and imaginary parts of S_1 are shown.

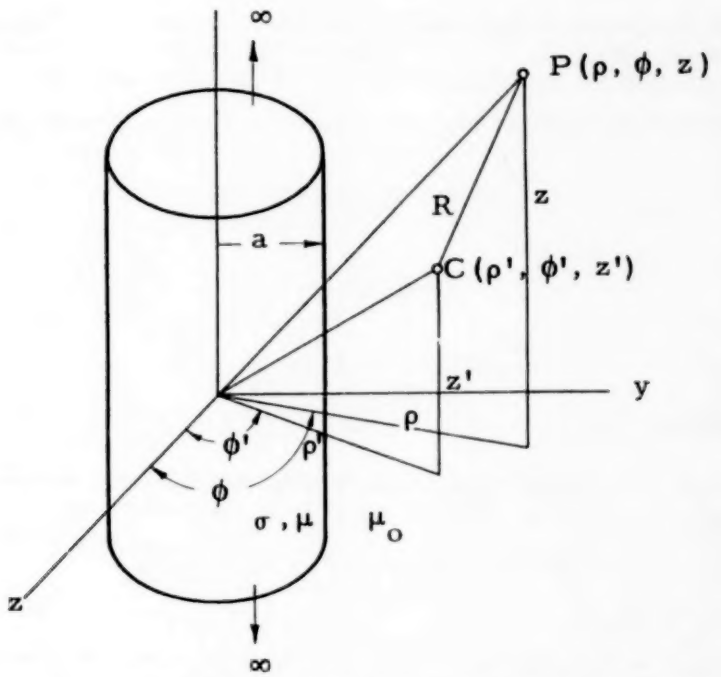

FIGURE 9.—*The cylindrical body and source C and observer P with the cylindrical coordinate system.*

External to the cylinder $(r>a)$

$$\vec{H}=-\text{grad}\ \Omega \tag{57}$$

where

$$\Omega=\Omega^p+\Omega^\infty. \tag{58}$$

For a magnetic charge at $C\ (\rho', \phi', z')$ and the observer at $P\ (\rho, \phi, z)$,

$$\Omega^p=\frac{\overline{K}}{4\pi R}, \tag{59}$$

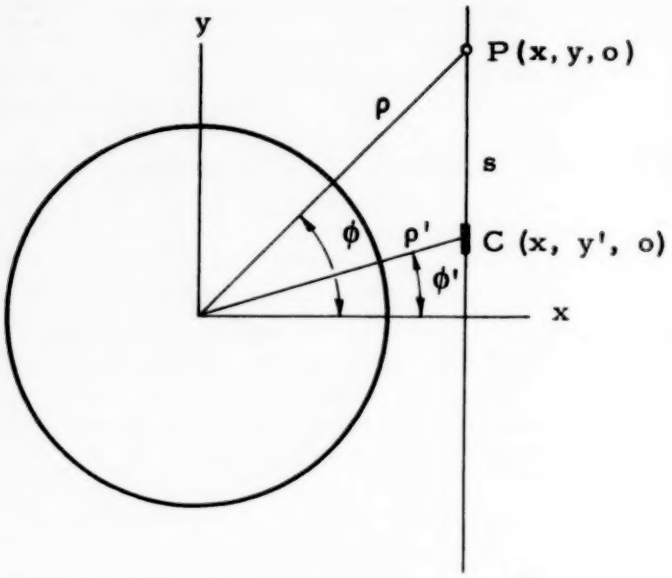

FIGURE 10.—*Special configuration of (dipole) source C and observer P when they are located in equatorial plane of cylinder.*

where R is the usual linear distance between C and P. Using a formula given by Watson [7] we find

$$\frac{1}{R}=\frac{1}{\pi}\int_{-\infty}^{+\infty}K_0(h\hat{\rho})e^{-ih(z-z')}dh, \tag{60}$$

where

$$R=\sqrt{(\hat{\rho})^2+(z-z')^2} \tag{61}$$

with

$$\hat{\rho}=[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')]^{\frac{1}{2}}. \tag{62}$$

Using the addition theorem

$$K_0(h\hat{\rho})=\sum_{m=0}^{\infty}\epsilon_m K_m(h\rho')I_m(h\rho)\cos m(\phi-\phi') \tag{63}$$

for $\rho<\rho'$ and with

$$\epsilon_0=1, \qquad \epsilon_m=2 \ (m=1, 2, 3 \ . \ . \ . \ ), \tag{64}$$

it now readily follows that

$$\Omega^p=\frac{\overline{K}}{4\pi}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_m(h\rho')\{I_m(h\rho)e^{-ih(z-z')}dh\}e^{-im(\phi-\phi')}. \tag{65}$$

This latter equation can be written symbolically in the manner

$$\Omega^p=\Gamma I_m(h\rho) \tag{66}$$

where Γ is the operator

$$\frac{\overline{K}}{4\pi^2}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_m(h\rho')\{\ .\ .\ .\ .\}e^{-ih(z-z')}dh\,e^{-im(\phi-\phi')}. \tag{67}$$

This suggests writing

$$\Omega^s=\Gamma A_m(h)K_m(h\rho) \tag{68}$$

or

$$\Omega=\Gamma[I_m(h\rho)+A_m(h)K_m(h\rho)] \tag{69}$$

for the external fields and

$$\Pi_z^* = \Gamma a_m(h) I_m(\alpha\rho), \tag{70}$$

$$\Pi_z = \Gamma b_m(h) I_m(\alpha\rho) \tag{71}$$

for the internal fields. The corresponding magnetic field components are

$$\left.\begin{aligned} H_\rho &= -\Gamma h[I_m'(h\rho) + A_m(h) K_m'(h\rho)] \\ H_\phi &= \Gamma \frac{im}{\rho}[I_m(h\rho) + A_m(h) K_m(h\rho)] \\ H_z &= \Gamma ih[I_m(h\rho) + A_m(h) K_m(h\rho)] \end{aligned}\right\} r \geqq a \tag{72}$$

and

$$\left.\begin{aligned} H_\rho &= \Gamma\left\{-ih\alpha a_m(h) I_m'(\alpha\rho) - \frac{\gamma^2 m}{\mu\omega\rho} b_m(h) I_m(\alpha\rho)\right\} \\ H_\phi &= \Gamma\left\{-\frac{hm}{\rho} a_m(h) I_m(\alpha\rho) - \frac{\gamma^2 a}{i\mu\omega} b_m(h) I_m'(\alpha\rho)\right\} \\ H_z &= \Gamma\{-\alpha^2 a_m(h) I_m(\alpha\rho)\} \end{aligned}\right\} r \leqq a \tag{73}$$

The unknown coefficients a_m, b_m, and A_m, which are functions of h, are determined from the boundary conditions. These are the normal flux, (i.e., μH_ρ) and the tangential fields, (i.e., H_ϕ and H_z) are continuous at $r=a$. The explicit result for $A_m(h)$ is found to be

$$A_m(h) = -\left[\frac{\tilde{I}_m(ha) - (\mu/\mu_1)\left[\tilde{I}_m(\alpha a) + \frac{m^2\gamma^2}{\alpha^4 h^2 a^4}\frac{1}{\tilde{I}_m(\alpha a)}\right]}{\tilde{K}_m(ha) - (\mu/\mu_0)\left[\tilde{I}_m(\alpha a) + \frac{m^2\gamma^2}{\alpha^4 h^2 a^4}\frac{1}{\tilde{I}_m(\alpha a)}\right]}\right]\frac{I_m(ha)}{K_m(ha)}, \tag{74}$$

where

$$\tilde{I}_m(\alpha) = \frac{I_m'(\alpha)}{\alpha I_m(\alpha)} \tag{75}$$

and

$$\tilde{K}_m(\alpha) = \frac{K_m'(\alpha)}{\alpha K_m(\alpha)}. \tag{76}$$

The preceding results can now be employed to calculate the magnetic field of a magnetic dipole. In the case of a y-directed dipole at C the potential at P is $-dl\partial\Omega/\partial y'$. The y-component of the field at P is obtained from

$$H_y = +dl\frac{\partial^2\Omega}{\partial y \partial y'}. \tag{77}$$

Restricting our attention again to the plane $z=z'=0$ and for $x=x'$ (see fig. 10), we have

$$H_y^p = -\frac{\overline{ik}dl}{2\pi s^2}, \tag{78}$$

where $s=y-y'>0$. Noting that

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial\rho} + \cos\phi\frac{\partial}{\rho\partial\phi}, \tag{79}$$

it readily follows that

$$H_y^s = \frac{\overline{K}dl}{4\pi^2}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_m(h)\left[hK_m'(h\rho)\sin\phi - \frac{im}{\rho}K_m(h\rho)\cos\phi\right] \times\left[hK_m'(h\rho)\sin\phi' + \frac{im}{\rho'}K_m(h\rho')\cos\phi'\right]e^{-ih(x-x')}dh e^{-im(\phi-\phi')}, \tag{80}$$

where $A_m(h)$ is given explicitly above.

5. A Related Cylindrical Problem

The above general solution for an infinitely long cylinder with dipolar excitation is rather complicated. The presence of the infinite integrals and the existence of Bessel functions of generally complex argument complicate the numerical aspects of the problem. It appears, however, that simplifications can be made under certain limiting conditions. For example, if the radius a is small and at low frequencies, ha is much less than unity over the important range of the integration. Thus,

$$A_m(h) \cong -\left[\frac{mKI_m(\gamma a)-(\gamma a)I'_m(\gamma a)}{mKI_m(\gamma a)+(\gamma a)I'_m(\gamma a)}\right]\frac{I_m(ha)}{K_m(ha)}, \tag{81}$$

where $K=\mu/\mu_0$. In the case of very high conductivity $|\gamma a|>>1$ this simplifies further to

$$A_m(h) \cong -\frac{I_m(ha)}{K_m(ha)}. \tag{82}$$

The square bracket term in equation (81) thus characterizes the frequency dependence of the response and is a function only of K and γa.

The secondary magnetic field H^s for the case of finite γa can then be written in terms of value of $\vec{H}^s$ for infinite γa in the following way (for a given mode number m)

$$\frac{\vec{H}^s}{[\vec{H}^s]_{\gamma a\to\infty}} \cong T_m \qquad (m=1,2,3,\ \ldots), \tag{83}$$

where

$$T_m=-\left[\frac{mKI_m(\sqrt{i}\,X)-\sqrt{i}\,XI'_m(\sqrt{i}\,X)}{mKI_m(\sqrt{i}\,X)+\sqrt{i}\,XI'_m(\sqrt{i}\,X)}\right] \tag{84}$$

and where $X=(\sigma\mu\omega)^{\frac{1}{2}}a$.

The total field is, of course, a sum over all modes, but because of the factor $I_m(ha)/K_m(ha)$, the relative response decreases rapidly with increasing value of m. This follows from the approximation

$$\frac{I_m(ha)}{K_m(ha)} \cong \frac{(ha)^{2m}}{(m!)\,(m-1)!\,2^{2m-1}}, \tag{85}$$

which is valid for $(ha)^2 \ll 1$.

To shed further light on the above approximate form, it is constructive to examine a related problem. If the source is an infinite line source of electric current I_0 located at (ρ_0, ϕ_0) and parallel to the z-axis, it is known that the secondary fields due to the presence of the cylinder are [10]

$$H^s_\rho=-\frac{I_0}{2\pi}\sum_{m=1}^{\infty} T_m \frac{a^{2m}}{\rho_0^m \rho^{m+1}} \sin\, m\,(\phi-\phi_0) \tag{86}$$

and

$$H^s_\phi=-\frac{I_0}{2\pi}\sum_{m=1}^{\infty} T_m \frac{a^{2m}}{\rho_0^m \rho^{m+1}} \cos\, m\,(\phi-\phi_0). \tag{87}$$

The structure of this exact two-dimensional solution is very similar to the approximated form of the three-dimensional solution mentioned above. Apparently if a is reasonably small, the induced currents flow mainly in the axial direction even for a localized excitation and thus the frequency dependence of the induced eddy currents are adequately described by the form of the solution of the corresponding two-dimensional problem.

The numerical values of the real and imaginary parts of the function T_m are illustrated in figure 11, where they are plotted as a function of X, the "relative radius" of the cylinder, for various values of the permeability ratio K. It is noted that the factor T_1, corresponding to the dominant mode, is very similar to the factor S_1 for the sphere shown in figure 8.

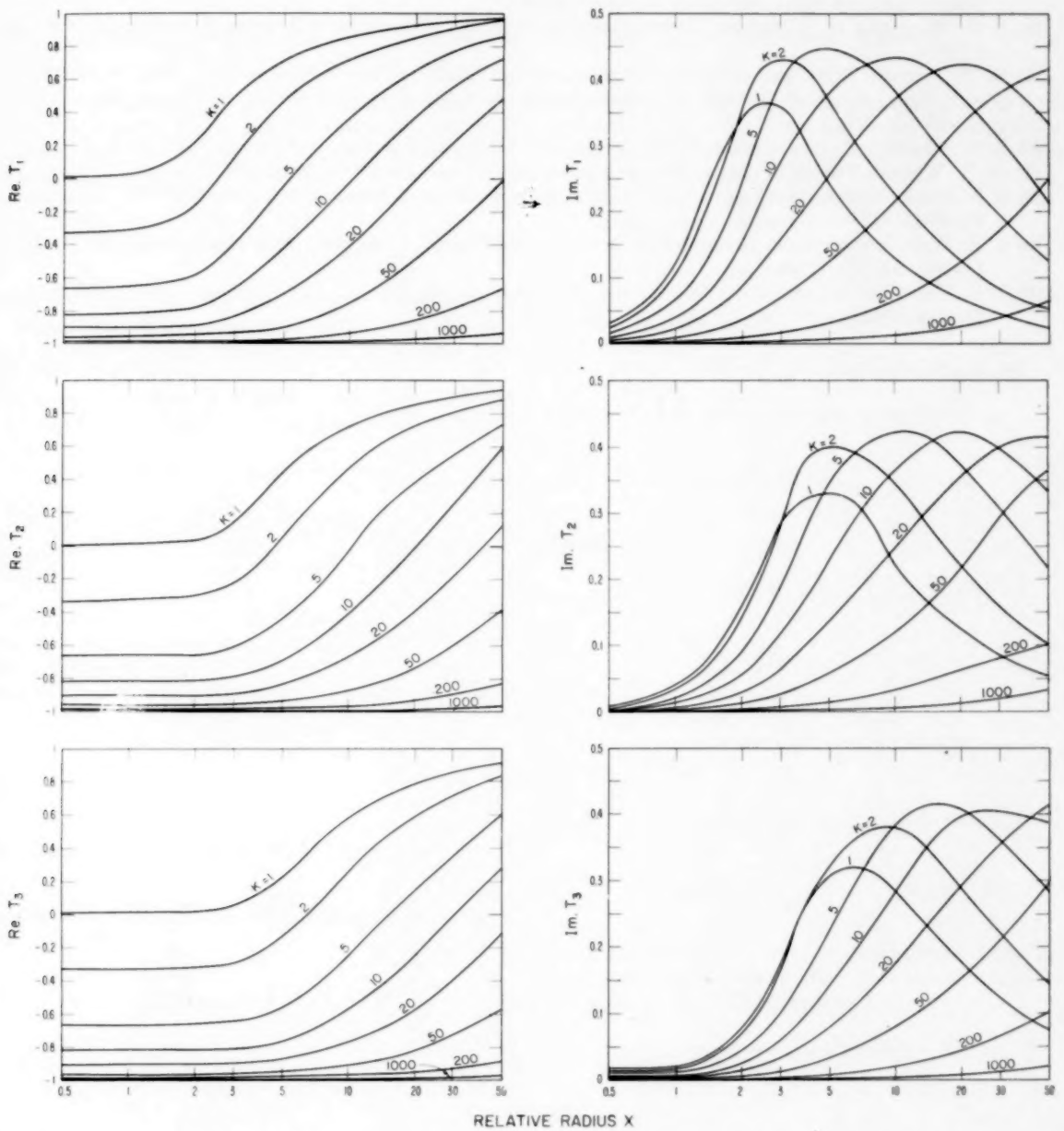

FIGURE 11.—*The variation of the response of a conducting cylinder as a function of* $X(=(\sigma\mu\omega)^{\frac{1}{2}}a)$ *for various permeability ratios* $K(=\mu/\mu_0)$.

The real and imaginary parts of T_1, T_2, and T_3 are shown.

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6. References

[1] C. A. Heiland, Geophysical exploration (Prentice-Hall, Inc., New York, N.Y., 1940).
[2] J. R. Wait, A conducting sphere in a time varying magnetic field, Geophysics **16,** 666 (1951).
[3] J. R. Wait, Mutual electromagnetic coupling of loops over a homogeneous ground, Geophysics **20,** 630 (1955); contains many references to related work.
[4] S. H. Ward and H. A. Harvey, Electromagnetic surveying of diamond drill holes, Can. Mining Manual, 1 (1954).
[5] W. R. Smythe, Static and dynamic electricity, 2d ed. (McGraw-Hill Book Co., New York, N.Y., 1950); such functions are tabulated in Tables of associated Legendre polynomials (Columbia Univ. Press, New York, N.Y.).
[6] S. A. Schelkunoff, Electromagnetic waves (D. Van Nostrand Co., Inc., New York, N.Y., 1943).
[7] G. N. Watson, Theory of Bessel functions (Cambridge Univ. Press, England, 1945).
[8] S. H. Ward, A method for measuring the electrical conductivity of diamond drill core specimens, Geophysics **18,** 434 (1953).
[9] J. R. Wait, A conducting permeable sphere in the presence of a coil carrying an oscillating current, Can. J. Phys. **31,** 670 (1953).
[10] J. R. Wait, The cylindrical ore body in the presence of a cable carrying an oscillating current, Geophysics **17,** 378 (1952).

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Kantorovich's Inequality[1]

Morris Newman

(August 18, 1959)

An elementary proof with a generalization of an inequality of Kantorovich is given.

Let A be a hermitian positive definite matrix with smallest eigenvalue α and largest eigenvalue β. Then Kantorovich's inequality [2] states that for all vectors x of unit norm,

$$(Ax, x)(A^{-1}x, x) \leq \frac{1}{4}\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\right\}^2. \tag{1}$$

In this note we give an elementary proof of (1) which allows an easy generalization. The results obtained do not depend on the order of A and this is accordingly left unspecified.

We assume only that A is hermitian, and that the eigenvalues of A are contained in the closed interval $m \leq t \leq M$.

Let $f(t)$, $g(t)$ be real functions such that

$$0 < f(t), g(t) < \infty \qquad \text{for} \quad m \leq t \leq M, \tag{2}$$

$$f(t), g(t) \quad \text{are convex} \qquad \text{for} \quad m \leq t \leq M. \tag{3}$$

Then (2) implies that the matrices $F=f(A)$, $G=g(A)$ are well-defined and are hermitian positive definite. We shall prove:

THEOREM. *Let x be any vector of unit norm. Put*

$$K=(Fx,x)(Gx,x).$$

Then for every positive c,

$$2K^{\frac{1}{2}} \leq max\left(cf(m)+\frac{1}{c}g(m), cf(M)+\frac{1}{c}g(M)\right). \tag{4}$$

If in addition $f(M)-f(m)$ and $g(M)-g(m)$ are of opposite sign, then

$$2K^{\frac{1}{2}} \leq rf(m)+\frac{1}{r}\,g(m), \tag{5}$$

where

$$r=\left(-\frac{g(M)-g(m)}{f(M)-f(m)}\right)^{\frac{1}{2}}.$$

PROOF. Since A is hermitian, A is unitarily equivalent to a diagonal matrix. We may therefore assume without loss of generality that $A=(a_i\delta_{ij})$, where $m \leq a_i \leq M$. Then also $F=(f(a_i)\delta_{ij})$ and $G=(g(a_i)\delta_{ij})$. Suppose that the i^{th} coordinate of x is x_i and set $t_i=|x_i|^2$, so that

$$t_i \geq 0, \qquad \sum t_i = 1, \tag{6}$$

x being of unit norm. Then K becomes

$$K=\sum f(a_i)t_i \sum g(a_i)t_i.$$

Let $c>0$ be arbitrary and rewrite K as

$$K=\sum cf(a_i)t_i \sum \frac{1}{c}\,g(a_i)t_i.$$

Then

$$2K^{\frac{1}{2}}=2\left(\sum cf(a_i)t_i\right)^{\frac{1}{2}}\left(\sum \frac{1}{c}\,g(a_i)t_i\right)^{\frac{1}{2}}$$

$$\leq \sum cf(a_i)t_i+\sum \frac{1}{c}\,g(a_i)t_i$$

$$\leq \max\left(cf(a_i)+\frac{1}{c}\,g(a_i)\right),$$

by (6).

We now consider the function

$$u=cf(t)+\frac{1}{c}g(t), \qquad m \leq t \leq M.$$

Because of assumptions (2) and (3) and the fact that c is positive, we have that u is convex and positive for $m \leq t \leq M$. This implies that

$$\max_{m \leq t \leq M} u = \max\left(u(m), u(M)\right)$$

which proves (4).

If in addition $f(M)-f(m)$ and $g(M)-g(m)$ are of opposite sign, the choice $c=r$ is permissible and makes

$$rf(m)+\frac{1}{r}g(m)=rf(M)\frac{1}{r}g(M),$$

[1] The preparation of this paper was supported (in part) by the Office of Naval Research.

[2] L. V. Kantorovich, Functional analysis and applied mathematics, Uspekhi Math. Nauk 3, 89 (1948).

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which gives (5). This completes the proof of the theorem.

Kantorovich's inequality (1) is the case $f(t)=t$, $g(t)=\frac{1}{t}$ for which $r=(mM)^{-\frac{1}{2}}$ (which is permissible when A is positive definite).

Ky Fan in a written communication to the author points out that the theorem can be generalized as follows:

Let A be a hermitian matrix of order n with all its eigenvalues contained in the closed interval $\alpha\leq t\leq\beta$. Let $x_1, x_2, \ldots, x_k$ be vectors in unitary n-space such that

$$\sum_{i=1}^{k}||x_i||^2=1.$$

Let $f_1, f_2, \ldots, f_m$ be positive convex functions for $\alpha\leq t\leq\beta$. If $c_1, c_2, \ldots, c_m$ are positive numbers satisfying $c_1c_2\ldots c_m=1$, then

$$\prod_{i=1}^{m}\left\{\sum_{j=1}^{k}(f_i(A)x_j,x_j)\right\}^{1/m}\leq\frac{1}{m}\max\left\{\sum_{i=1}^{m}c_if_i(\alpha),\sum_{i=1}^{m}c_if_i(\beta)\right\}.$$

Furthermore if A is positive definite and $\alpha>0$, Kantorovich's inequality can be replaced by

$$\left\{\sum_{i=1}^{k}(Ax_i,x_i)\right\}\left\{\sum_{i=1}^{k}(A^{-1}x_i,x_i)\right\}\leq\frac{1}{4}\left\{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}}+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}}\right\}^2.$$

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A Symmetric Continuous Poker Model

A. J. Goldman and J. J. Stone [1]

(August 18, 1959)

Beginning with von Neumann, mathematicians concerned with the rational analysis of conflict situations have realized that investigation of accurate or simplified versions of common card games leads to techniques and insights applicable to situations of military or economic interest. In the present paper, a symmetric poker model one stage more complicated than the original von Neumann game is solved. There are two bet levels, a, b, and an ante of 1 unit ($a>b>1$); no raises are permitted. The game has a unique optimal strategy, which forbids bluffing on a low hand. The limiting case $b=1$ is shown to yield the von Neumann model.

1. Introduction

The technical literature of the mathematical theory of games of strategy can be roughly subdivided into (*i*) *general* papers, which deal with concepts, existence theorems and computational methods relevant for broad classes of games, and (*ii*) *specific* papers, in which the solutions of particular games are determined. Although many of the studies in the second category have dealt with problems suggested by economic or military conflicts, e.g. [*1*, *2*] [2], a substantial fraction have dealt with accurate or simplified versions of common card games, the best known example being von Neumann's analysis of a simple poker model ([*3*], sec. 19.4 to 19.10). Bridge and baccarat have been treated [*4*, *5*], but poker has remained the favorite topic for these studies [*3*, *6* to *10*], perhaps because the poker models investigated have presented just the right degree of challenge ("not easy, but not impossibly hard") to the ingenuity of the analyst. Of course the techniques and insights developed in these investigations are then applicable to situations of greater practical importance.

The present paper deals with a symmetric poker game one stage more complicated than von Neumann's original model. The game is *continuous* in the sense that the difficulties [*6*] due to the enormous but finite number of possible hands have been avoided by assuming (as von Neumann did) an infinite continuum of possible hands. The (more difficult) asymmetric version has also been solved by the authors; the results will appear elsewhere [*11*].

2. Description of the Game

The rules of the game are as follows. The two players R and S first *ante* 1 unit. They then receive independent random numbers (*hands*) from the interval [0, 1]. Each player knows his own hand, but not that of his opponent. The players act simultaneously, each either *dropping*, betting an additional $b-1$ units, or betting an additional $a-1$ units. If both drop, then play ends and no payment is made. If both bet $b-1$ units or both bet $a-1$ units, then play ends and the player with the higher hand [3] wins the ante and bet of his opponent. If one player bets $a-1$ units and the other bets $b-1$, then the latter can either *fold* (in which case he loses his ante and previous bet to his opponent) or *see* [4] by betting an additional $a-b$ units (in which case the player with the higher hand wins the ante and previous bet or bets of his opponent); in either case play ends. Naturally we assume $a>b>1$.

Each player has four courses of action for any given hand. These are (1) to drop, (2) to bet b–1 units with the intention of folding if the opponent bets a–1 units, (3) to bet b–1 units with the intention of seeing if the opponent bets a–1 units, and (4) to bet a–1 units. Figures 1 and 2 show the algebraic payoff from player S to player R for any choices of courses of action by the players.

A *strategy* (for either player) is a quadruple $r(z)=\{r_i(z)\}$ $(i=1,2,3,4)$ of functions defined for $0\leq z\leq 1$. Since $r_i(z)$ represents the probability [5] that the player will choose the ith course of action when his hand is z, the restrictions

$$r_i(z)\geq 0, \qquad r_1(z)+r_2(z)+r_3(z)+r_4(z)=1$$

are imposed. The "smoothness" conditions on these functions are (*i*) that they be integrable, (*ii*) that if $r_i(z')>0$ then, for any interval J of which z' is an interior [6] point,

$$\int_J r_i(t)dt>0,$$

and finally [7] (*iii*) that

$$\int_0^z [br_2(t)-ar_4(t)]dt=0 \qquad \text{on} \quad [(b-1)/b, (a-1)/a]$$

[1] Present address, Stanford University.
[2] Italicized figures in brackets indicate the literature references at the end of this paper.

[3] The possibility of equal hands has zero probability and so can be disregarded.
[4] This possibility distinguishes our model from the case $m=3$ of the version studied in part II of [*6*].
[5] More precisely, $r_i(z)$ is a probability density function.
[6] The words "interior" and "open" are to be understood relative to the interval [0,1].
[7] This very weak but *ad hoc* condition is precisely what is needed to prove (b) and (c) of lemma 8.

implies

$$br_2(z)-ar_4(z)=0 \qquad \text{on} \quad [(b-1)/b,\,(a-1)/a]$$

It is convenient to define, for any $r(z)$,

$$R_i=\int_0^1 r_i(t)dt.$$

For any $r(z)$, $G_i(z)$ will denote the expected payoff to player R, using $r(z)$, against a fixed hand z and a fixed course of action i of his opponent. Thus the expected payoff function of the game is given by

$$K(r,r^*)=\int_0^1 \sum r_i^*(z)G_i(z)dz \tag{2.1}$$

The explicit formulas [8] for the $G_i(z)$ are readily found from figures 1 and 2:

R \ S	1	2	3	4
1	0	−1	−1	−1
2	1	b	b	$-b$
3	1	b	b	a
4	1	b	a	a

FIGURE 1

R has the higher hand.

R \ S	1	2	3	4
1	0	−1	−1	−1
2	1	$-b$	$-b$	$-b$
3	1	$-b$	$-b$	$-a$
4	1	b	$-a$	$-a$

FIGURE 2

S has the higher hand.

$$G_1(z)=\int_0^z [r_2(t)+r_3(t)+r_4(t)]dt + \int_z^1 [r_2(t)+r_3(t)+r_4(t)]dt, \tag{2.2}$$

$$G_2(z)=\int_0^z [-r_1(t)-br_2(t)-br_3(t)+br_4(t)]dt + \int_z^1 [-r_1(t)+br_2(t)+br_3(t)+br_4(t)]dt, \tag{2.3}$$

$$G_3(z)=\int_0^z [-r_1(t)-br_2(t)-br_3(t)-ar_4(t)]dt + \int_z^1 [-r_1(t)+br_2(t)+br_3(t)+ar_4(t)]dt, \tag{2.4}$$

$$G_4(z)=\int_0^z [-r_1(t)-br_2(t)-ar_3(t)-ar_4(t)]dt + \int_z^1 [-r_1(t)-br_2(t)+ar_3(t)+ar_4(t)]dt. \tag{2.5}$$

[8] Note that the $G_i(z)$'s are shown by these formulas to be *continuous* functions.

3. Plan of Attack

It would not be very instructive simply to state the solution of the game and demonstrate its correctness. Instead, a slightly polished version of the process actually used to solve the game will be presented. This is done in sections 4 and 5, in which it is proved (subject to the plausible assumption stated in sec. 4) that the only possible optimal strategy is the one obeying the conditions (a) to (d) of lemma 8, which deal with the behavior of the strategy on three intervals $[0,g)$, (g,g'), and $(g',1]$ which might be thought of as the sets of *low* hands, *intermediate* hands, and *high* hands. It is then relatively easy to verify that this particular strategy is indeed the unique optimal strategy; this is done in section 6, primarily by inspection of the expected payoff function (2.1). In section 7 the limiting case $b=1$ is dealt with, and this game is found to have a unique optimal strategy obtained [9] by "passing to the limit" in the optimal strategy for $b>1$. The von Neumann poker game of sections 19.4 to 19.10 of [3] is shown to arise as a "reduced game" from this limiting case.

In section 4 it is shown that $r_3(z)=0$ on $[0,1]$ for every optimal $r(z)$. The points g, g' are located, and the unique optimal $r(z)$ is isolated in section 5. The arguments of these sections are essentially all applications of the following *domination criterion:*

The strategy $r(z)$ is optimal only if the following is true: For each z, i for which $G_i(z)$ (as a function of i) does not assume its minimum, $r_i(z)=0$.

The correctness of this criterion follows from the smoothness condition (*ii*) and the fact that $r(z)$ is optimal only if

$$K(r,r)=\min_{r^*} K(r,r^*).$$

4. Elimination of One Course of Action

The elimination is effected as follows.

LEMMA 1. *For every optimal $r(z)$, $r_3(z)=0$ on $[0,1]$.*

Otherwise, for some optimal $r(z)$, $R_3>0$ and $r_3(z)>0$ for some $z>0$. If z' is the least upper bound of the set of all such z, then by (2.4) and (2.5)

$$G_4(z')-G_3(z')=-(a-b)R_3-2b\int_{z'}^1 r_2(t)dt<0,$$

so that by continuity and the domination criterion $r_3(z)=0$ in a neighborhood of z', contradicting the definition of z'.

The following plausible assumption (which will be verified in sec. 6) is now made:

ASSUMPTION: *For every optimal $r(z)$, $r_4(1)>0$ (and thus $R_4>1$).*

[9] This is not quite the case for the asymmetric version see [11].

5. Derivation of the Necessary Conditions

Since application of the domination criterion requires examining the signs of the pairwise differences of the $G_i(z)$'s, the formulas for these differences (based on (2.2) to (2.5) and the condition $r_3(z)=0$ on [0,1]) will first be given:

$$G_2(z)-G_1(z)=(b-1)-bR_1-2b\int_0^z r_2(t)dt \quad (5.1)$$

$$G_3(z)-G_1(z)=bR_2+aR_4-1-2b\int_0^z r_2(t)dt -2a\int_0^z r_4(t)dt \quad (5.2)$$

$$G_3(z)-G_2(z)=(a-b)R_4-2a\int_0^z r_4(t)dt \quad (5.3)$$

$$G_4(z)-G_1(z)=-1-bR_2+aR_4-2a\int_0^z r_4(t)dt \quad (5.4)$$

$$G_4(z)-G_2(z)=-2bR_2+(a-b)R_4-2b\int_0^z r_2(t)dt -2a\int_0^z r_4(t)dt \quad (5.5)$$

$$G_4(z)-G_3(z)=-2bR_2+2b\int_0^z r_2(t)dt \quad (5.6)$$

The first four of these continuous functions are clearly monotone nonincreasing. The following table[10] defines the maximal open intervals in which these functions are positive (+), zero (0), and negative (−). For each interval in the (+) and (−) columns, the result of applying the domination criterion is given. Inspection of (5.1) to (5.4) reveals conditions on the intervals in the (0) column which are given below the interval. A question mark means that the interval may (so far as is known at the moment) be vacuous or degenerate to a single point; the condition written below such an interval is to apply only if the interval is non-degenerate.

The next three lemmas yield the values of R_1, R_2 and R_4 for optimal $r(z)$ (lemma 1 shows that $R_3=0$).

LEMMA 2. *For every optimal $r(z)$:*

(a) $G_2(0)-G_1(0)=(b-1)-bR_1\geq 0.$

(b) $G_4(0)-G_1(0)=-1-bR_2+aR_4\leq 0.$

If (a) were false, then the monotonicity of $G_2(z)-G_1(z)$ and the domination criterion would imply $R_1=0$, so that (a) would be true. If (b) were false, then since $G_4(1)-G_1(1)<0$ one would have

$$G_4(z')-G_1(z')=0$$

for some z' in (0,1); by (5.10) $r_4(z)=0$ on $[0,z')$; so by (5.4)

$$G_4(0)-G_1(0)=G_4(z')-G_1(z')=0$$

and (b) would be true.

LEMMA 3. *For every optimal $r(z)$:*

(a) $G_2(0)-G_1(0)=(b-1)-bR_1=0.$

(b) $G_4(0)-G_1(0)=-1-bR_2+aR_2=0.$

If $G_2(1)-G_1(1)>0$. then the monotonicity of $G_2(z)-G_1(z)$ and the domination criterion would imply $R_2=0$, so that on the one hand the initial assumption would read

$$(b-1)-bR_1>0,$$

[10] Of course lemma 1 renders superfluous the appearance of $r_3(z)=0$ in the table.

Difference	(+)	(0)	(−)	
$G_2(z)-G_1(z)$	$[0,g_-^{21})$? $r_2(z)=0$	(g_-^{21},g_+^{21})? $r_2(z)=0$	$(g_+^{21},1]$? $r_1(z)=0$	(5.7)
$G_3(z)-G_1(z)$	$[0,g_-^{31})$? $r_3(z)=0$	(g_-^{31},g_+^{31})? $r_1(z)=1$	$(g_+^{31},1]$ $r_1(z)=0$	(5.8)
$G_3(z)-G_2(z)$	$[0,g_-^{32})$ $r_3(z)=0$	(g_-^{32},g_+^{32})? $r_4(z)=0$	$(g_+^{32},1]$ $r_2(z)=0$	(5.9)
$G_4(z)-G_1(z)$	$[0,g_-^{41})$? $r_4(z)=0$	(g_-^{41},g_+^{41})? $r_4(z)=0$	$(g_+^{41},1]$ $r_1(z)=0$	(5.10)

and on the other hand $R_1=1-R_4$ (by lemma 1, using $R_2=0$). Combining these statements yields $R_4>1/b$, which together with (5.4) and $R_2=0$ yields

$$G_4(0)-G_1(0)=-1+aR_4>0,$$

contradicting (b) of lemma 2. Thus $G_2(1)-G_1(1)\leq 0$, which together with (a) of lemma 2 implies that

$$G_2(z')-G_1(z')=0$$

for some z' in [0,1]. By (5.7) $r_2(z)=0$ on $[0,z')$; so by (5.1),

$$G_2(0)-G_1(0)=G_2(z')-G_1(z')=0,$$

and (a) is proved. If (b) were false, then by (b) of lemma 2, the monotonicity of $G_4(z)-G_1(z)$, and the domination criterion, we would have $R_1=0$, contradicting (a).

LEMMA 4. *For every optimal $r(z)$:*

(a) $R_1=(b-1)/b$.

(b) $R_2=(a-b)/b(a+b)$.

(c) $R_4=2/(a+b)$.

This follows from lemma 3 and the corollary $R_1+R_2+R_4=1$ of lemma 1.

Next the points g, g' will be located, and the behavior of an optimal $r(z)$ on $[0,g)$, (g,g') and $(g',1]$ determined.

LEMMA 5. *For every optimal $r(z)$:*

(a) $g_+^{21}=g_+^{41}=g=(b-1)/b$.

(b) $r_1(z)=1$ *on* $[0,g)$.

(c) $r_1(z)=0$ *on* $(g,1]$.

By (5.5) and lemma 4,

$$G_4(z)-G_2(z)=2b\int_0^z r_2(t)dt-2a\int_0^z r_4(t)dt \quad (5.11)$$

If $g_+^{21}<g_+^{41}$, then by (5.7) and (5.10) one has $r_2(z)=1$ on (g_+^{21}, g_+^{41}), $r_1(z)=1$ on $[0, g_+^{21})$, and $r_4(z)=0$ on $[0, g_+^{41})$, so that the last equation yields

$$G_4(g_+^{41})-G_2(g_+^{41})>0;$$

continuity and the domination criterion show that $r_4(z)=0$ in a neighborhood of g_+^{41}, but (5.4) and (b) of lemma 3 show that this contradicts the definition of g_+^{41}. An entirely similar argument refutes the possibility $g_+^{41}<g_+^{21}$, and so $g_+^{21}=g_+^{41}$. By (5.7) and (5.10) we have $r_1(z)=1$ on $[0, g_+^{21})$ and $r_1(z)=0$ on $(g_+^{21}, 1]$, so that $g_+^{21}=R_1$; an application of (a) of lemma 4 completes the proof.

LEMMA 6. *For every optimal $r(z)$:*

(a) $g_+^{32}=g'=(a-1)/a$.

(b) $r_4(z)=1$ *on* $(g', 1]$.

Note first that $r_4(z)=1$ on $(g_+^{32}, 1]$ since (i) $r_2(z)=0$ on $(g_+^{32}, 1]$ by (5.9), (ii) $r_1(z)=0$ on $(g, 1]$ by lemma 5, and (iii) $g<g_+^{32}$ because $G_3(g)-G_2(g)>0$ by (5.3) and the fact (see (5.10)) that $r_4(z)=0$ on $[0, g)$. It follows that

$$\int_0^{g_+^{32}} r_4(t)dt+(1-g_+^{32})=R_4=2/(a+b).$$

The definition of g_+^{32} together with (5.3) shows that

$$\int_0^{g_+^{32}} r_4(t)dt=(a-b)R_4/2a=(a-b)/a(a+b), \quad (5.12)$$

and combining the last equations yields the asserted value of g_+^{32}.

LEMMA 7. *For every optimal $r(z)$,*

$$G_4(z)-G_2(z)=0 \qquad \text{on } [g, g'].$$

Since $r_2(z)=0$ on $(g', 1]$, (5.11) and (5.12) imply

$$G_4(g')-G_2(g')=2bR_2-(a-b)R_4=0,$$

where the last equality follows from lemma 3. Let z^* be a point at which $G_4(z)-G_2(z)$ attains its maximum M on $[g,g']$; the last equation shows that $M\geq 0$. If $M>0$, then by continuity and the domination criterion one has $r_4(z)=0$ (and thus $r_2(z)=1$, since $r_1(z)=0$ on $(g,1]$) on some interval (z^*,z'); by (5.5) $G_4(z)-G_2(z)$ is strictly increasing on (z^*,z'), which contradicts the definition of z^* unless $z^*=g'$, in which case $M=0$ by the last equation. Thus $M=0$; an entirely similar argument shows that the minimum value of $G_4(z)-G_2(z)$ on $[g,g']$ is $m=0$, and so the lemma is proved.

The completion and summary of the preceding material is as follows.

LEMMA 8. *Let $g=(b-1)/b$ and $g'=(a-1)/a$. If the assumption at the end of section 4 is correct, then the only possible optimal strategy is the strategy defined by*

(a) $r_1(z)=1$ *on* $[0,g)$.

(b) $r_2(z)=a/(a+b)$ *on* (g,g').

(c) $r_4(z)=b/(a+b)$ *on* (g,g').

(d) $r_4(z)=1$ *on* $(g',1]$.

From lemma 1, lemma 5, and lemma 6 one has (a), (d), and also

$$r_2(z)+r_4(z)=1 \qquad \text{on } (g,g'). \quad (5.13)$$

From lemma 7, (5.5), and lemma 4 one has

$$\int_0^z [br_2(t)-ar_4(t)]dt=0 \qquad \text{on} \quad (g,g'),$$

which by smoothness condition (*iii*) (see section 2) implies

$$br_2(z)-ar_4(z)=0 \qquad \text{on } (g,g');$$

this and (5.13) imply (b) and (c).

6. Proof That the Conditions Are Necessary and Sufficient for Optimality

In this section the notation $r(z)$ is reserved for the particular strategy described in lemma 8, while arbitrary strategies are denoted $r^*(z)$.

LEMMA 9. *$K(r,r_0^*)=\min_{r^*} K(r,r^*)$ if and only if $r_0^*(z)$ obeys*

(a) $r_3^*(z)=0$ on $[0,g)$.

(b) $r_1^*(z)=r_3^*(z)=0$ on (g,g').

(c) $r_1^*(z)=r_2^*(z)=0$ on $(g',1]$.

The $K(r,r^*)$ of (2.1) is first written as the sum of these integrals:

$$\int_0^g [G_1(z)+r_2^*(z)(G_2(z)-G_1(z))+r_3^*(z)(G_3(z)-G_1(z))+r_4^*(z)(G_4(z)-G_1(z))]dz.$$

$$\int_g^{g'} [G_2(z)+r_1^*(z)(G_1(z)-G_2(z))+r_3^*(z)(G_3(z)-G_2(z))+r_4^*(z)(G_4(z)-G_2(z))]dz.$$

$$\int_{g'}^1 [G_4(z)+r_1^*(z)(G_1(z)-G_4(z))+r_2^*(z)(G_2(z)-G_4(z))+r_3^*(z)(G_3(z)-G_4(z))]dz.$$

It is easily verified that $r(z)$ has the properties described in lemma 4, and since $r_3(z)=0$ on $[0,1]$, $r(z)$ obeys (5.1)—(5.6). Using these facts, one finds that on $[0,g)$ the coefficients of $r_2^*(z)$ and $r_4^*(z)$ in the first integral vanish, while that of $r_3^*(z)$ is positive. Hence $r_0^*(z)$ minimizes the first integral if and only if it satisfies (a). Similarly, $r_0^*(z)$ minimizes the second integral if and only if it satisfies (b), and minimizes the third integral if and only if it satisfies (c).

THEOREM. *$r(z)$ is the unique optimal strategy.*

Since $r(z)$ obeys (*a*) to (*c*) of lemma 9,

$$K(r,r)=\min_{r^*} K(r,r^*)$$

which (since the game is symmetric) implies that $r(z)$ is optimal. Thus *any* optimal $r_0^*(z)$ must satisfy

$$K(r,r_0^*)=\min_{r^*} K(r,r^*)$$

and thus, by lemma 1 and (c) of lemma 9, must satisfy $r_4^*(1)=1$. Hence the assumption at the end of section 4 is correct, and so, by lemma 8, $r(z)$ is the *only* optimal strategy.

7. Discussion of a Limiting Case

The limiting case $b=1$ will now be briefly discussed. (Recall that $b>1$ previously.) It is easily verified that the results of the previous sections remain valid (many of the proofs can be considerably simplified), so that the game with $b=1$ has a unique optimal strategy which is obtained from that for $b>1$ (i.e., from (a) to (d) of lemma 8) by setting $b=1$. Since $g=0$ for $b=1$, the unique optimal strategy is given by

(a) $r_2(z)=a/(a+1)$ on $[0,(a-1)/a)$.

(b) $r_4(z)=1/(a+1)$ on $[0,(a-1)/a)$

(c) $r_4(z)=1$ on $((a-1)/a,1]$.

Upon changing

player R, player S, r_2, r_3, r_4, a

to player 1, player 2, ρ_3, ρ_2, ρ_1, a/b

respectively, one finds that this strategy is precisely the unique optimal strategy of the continuous von Neumann poker game discussed in sections 19.4 to 19.10 of [3]. To explain this, note that (2.2) and (2.3), with $b=1$, yield

$$G_2(z)-G_1(z)=-R_1-2\int_0^z [r_2(t)+r_3(t)]dt\leq 0$$

for all strategies $r(z)$, which on the one hand shows that the first course of action is dominated, and on the other hand yields a quick proof[11] that $r_1(z)=0$ on $[0,1]$ for all optimal $r(z)$. Thus, without loss of optimal strategies, the game can be "reduced" by "striking out" the first course of action for both players. This reduced game is easily seen to be identical with the von Neumann game, except for some changes of language and a multiplicative factor b in the expected payoff function. Thus the von Neumann game appears as a reduced form of the limiting case $b=1$ of the game solved in this paper.

[11] If $r_1(z')>0$ for some z', then $R_1>0$; the formula and the domination criterion yield $r_1(z)=0$ for all z, a contradiction.

8. References

[1] L. E. Dubins, A discrete evasion game, Ann. Math. Study **39,** 231-255 (1957).

[2] M. Shubik, Edgeworth market games, Ann. Math. Study **40,** 267-278 (1959).

[3] J. von Neumann and O. Morgenstern, Theory of games and economic behavior (3d ed., Princeton Univ. Press, Princeton, N.J., 1953).

[4] G. L. Thompson, Bridge and signalling, Ann. Math. Study **28,** 279-289 (1953).

[5] J. G. Kemeny, L. Snell, and G. L. Thompson, Game-theoretic solution of baccarat, Am. Math. Monthly **64,** 465-469 (1957).

[6] D. B. Gillies, J. P. Mayberry, and J. von Neumann, Two variants of poker, Ann. Math. Study **28,** 13-5 (1953).

[7] H. Kuhn, A simplified two-person poker, Ann. Math. Study **24,** 97-103 (1950).

[8] J. Nash and L. S. Shapley, A simple three-person poker game, Ann. Math. Study **24,** 105-116 (1950).

[9] R. Bellman, On games involving bluffing, Rend. Circ. Mat. Palermo [**2**] **1,** 139-156 (1952).

[10] S. Karlin and R. Restrepo, Multistage poker models, Ann. Math. Study **39,** 337-363 (1957).

[11] A. J. Goldman and J. J. Stone, A continuous poker game, to appear in Duke Math. J. (1960).

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Moebius Function on the Lattice of Dense Subgraphs

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The Moebius function f_k on the lattice of k-dense subgraphs of a connected graph, defined in a previous paper, is calculated for graphs G containing isthmuses and articulators. f_1 evaluated for the null graph ϕ is shown to vanish if G contains an isthmus, while for any integer q there exist graphs containing articulators for which $f_1(\phi)=q$. The "lattice of path sets" joining a pair of points and the lattice of graphs "associated with G and a subgraph G'" are defined and the Moebius functions on these lattices are shown in certain cases to be related to f_1.

The concepts "k-dense subgraph," "isthmus," and "articulator" were defined in a previous paper [1],[1] where it was shown that the k-dense subgraphs of a given connected graph G, together with the null graph ϕ, form a lattice. The Moebius function f_k on this lattice will be defined in the present paper, and $f_k(\phi)$ and, in particular, $f_1(\phi)$ will be evaluated for various types of graphs. It is found that $f_1(\phi)=0$ if G contains an isthmus, and $|f_1(\phi)|\leq 1$ unless G contains an articulator. However, for every integer q, there exists a graph containing an articulator for which $f_1(\phi)=q$. A second lattice is formed by the sets of paths joining a given pair of points and containing among them all the points of G, and when G contains an articulator in which there are just two points, $f_1(\phi)$ can be expressed in terms of the Moebius function defined on the second lattice. The graphs "associated with a graph G and a subgraph G'" are defined and shown to form a lattice. A relation is found between the Moebius function on this lattice and the functions $f_1(\phi)$ defined for the individual lattice elements.

1. Definitions

A linear, undirected *graph* G is a set of elements, called points, together with a set of ordered pairs of these elements which define a symmetric nonreflexive, binary relation. For any given graph G, the number of points will be assumed finite and unless otherwise specified this number will be denoted by the symbol n. A subset of the set of points in G, together with all the ordered pairs which contain only points in the subset, is called a *subgraph*. The *union*, *intersection*, or *difference* of two subgraphs is the subgraph determined respectively by the union, intersection or difference of the sets of points in the two subgraphs. If G' is a subgraph of G, we say G' *is contained in* G and write $G'\subset G$. Two points appearing in the same ordered pair in G are said to be *neighbors*. For any given pair of points, p and q, a sequence of distinct points $\{p_1, \ldots, p_r\}$, with the property that $p_1=p$, $p_r=q$, and p_i and p_{i+1} are neighbors for $1\leq i<r$, is called a *path* joining p and q, in which p_1 is the *initial point* and p_r the *terminal point*. A path containing r points will be called an *r-cycle* if its initial and terminal points are neighbors and no point of the path is a neighbor of more than two other points of the path. If the sequence of points of a path P contains a proper subsequence which is also a path, the subsequence is called a *subpath* contained in P. Two points joined by a path are said to be *connected*. A graph is *connected* if it has only one point, or more than one point and each pair of points connected. If every pair of points are neighbors, G is said to be *completely connected*. A graph which not connected is *disconnected*. The null graph ϕ is disconnected. Unless otherwise specified, the symbol G will denote a connected graph.

If G′ is a subgraph and G–G' is not connected, G' is said to *disconnect* G. If G' disconnects G and G–G' contains, and is contained in, a set of connected graphs, the union of any pair of which is not connected, this set is called the *partition of* G–G'. If G' contains precisely m points, the partition of G–G' consists of at least $k+1$ connected graphs, and G' contains no proper subgraph which disconnects G, then G' is called an $[m, k]$-isthmus if it is completely connected and an $[m, k]$-articulator if it is not completely connected. An $[m, 1]$-articulator or $[m, 1]$-isthmus will be called an *articulator* or *isthmus* respectively when the number of points is not relevant.

For any subgraph G', $G(G')$ will denote the subgraph determined by the set of all points in G–G' which have neighbors in G'. If G' is a single point p, we shall denote this set by $G(p)$. A connected subgraph G' is said to be *k-dense* provided there are at most k-1 points in G–G' which are not points of $G(G')$. A k-dense subgraph which contains no k-dense proper subgraph is *D_k-minimal*. A k-dense

[1] Figures in brackets indicate the literature references at the end of this paper.

proper subgraph of G, which is contained in no other k-dense subgraph except G itself, is called *D_k-maximal*. The symbol Γ_k will denote the union of all the D_k-minimal subgraphs of G.

Let S_k denote the set of all the k-dense subgraphs of G which contain at least k points, together with the null graph. It has been shown in a previous paper [1, theorem 2.2] that the set S_k form a lattice under the relation of set inclusion, in which the l.u.b. of two subgraphs is their union; and the g.l.b. of two subgraphs is their intersection, if it is in S_k, and otherwise is either ϕ or a graph in the partition of the intersection, if the latter is not connected. The lattice formed by the graphs in S_k will be called the *lattice of k-dense subgraphs*. For any finite lattice composed of elements partially ordered by the relation $\geqslant$ there is a greatest element I such that $I \geqslant x$ for every element x [2, ch. II]. The *Moebius function* on the lattice is a relation which associates with every lattice element x a unique integral number $M(x)$, defined by

$$M(I)=1$$

$$M(x)=-\sum_{y>x} M(y)$$

For any lattice element x, $M(x)$ given by this definition is equal to the number given for the same element by the usual definition of the Moebius function [2, ch. I] on the dual lattice obtained by replacing $\geqslant$ by $\leqslant$. On the lattice of subgraphs in S_k for which G is the greatest element and $\geqslant$ is understood to mean $\supseteq$, the Moebius function will be denoted by f_k and the number which it associates with a given subgraph G' by $f_k(G')$. A second function, defined on the entire set of non-null subgraphs of G is the *size* σ. For any subgraph $G' \neq \phi$, $\sigma(G')$ is the number of points in G'.

2. Completely Connected Graphs and Graphs Containing Isthmuses

The Moebius function on the lattice of k-dense subgraphs will be shown to depend on the size of G and on the way in which the points are connected. First, we shall prove some preliminary combinatorial theorems.

LEMMA 2.1. *If G_1 is a subgraph containing a proper 1-dense subgraph G_2 and S is the set of all subgraphs of G_1 which are 1-dense and contain G_2, then*[2]

$$\sum_{G' \in S}(-)^{\sigma(G')}=0.$$

It has been proved [3, lemma 2.1] that a subgraph which contains a 1-dense subgraph is 1-dense. Therefore a subgraph in S is formed by taking G_2 alone or the union of G_2 and the subgraph determined by any set of points from G_1-G_2. If $\sigma(G_1)-\sigma(G_2)=m$, the number of subgraphs in G_1-G_2 which have k points is $\binom{m}{k}$ and so

$$\sum_{G' \in S}(-)^{\sigma(G')}=(-)^{\sigma(G_2)}\sum_{k=0}^{m}\binom{m}{k}(-)^k=(-)^{\sigma(G_2)}(1-1)^m=0.$$

A sequence of subgraphs $\{H_1, \ldots, H_q\}$, for any positive integer q, will be called an *H-sequence* if, and only if, it has the following properties: (1) H_1 is connected; (2) for every $i>1$, $H_i \subset G(H_{i-1}) - \sum_{j=1}^{i-2} G(H_j) - H_1$. If $H_1=G'$, for a given connected subgraph G', the *H-sequence* is said to be *based on G'*.

LEMMA 2.2. *If $\{H_1, \ldots, H_q\}$ is an H-sequence, then:*

1. *For any $r \geq 1$, $\sum_{i=1}^{r} H_i$ is connected.*
2. *If $q>1$ and $\sum_{i=1}^{q} H_i$ is 1-dense, then either there exists a non-null subgraph H_{q+1} such that $\{H_1, \ldots, H_q, H_{q+1}\}$ is an H-sequence, or $\sum_{i=1}^{q} H_i$ is 1-dense.*

To prove part 1, suppose p_k is a point in H_k for $k \leq r$. Then p_k has a neighbor p_{k-1} in H_{k-1}, and given a point p_i in H_i for $i \leq k-1$, p_i has a neighbor p_{i-1} in H_{i-1}. Thus there exists a path $p=\{p_k, \ldots, p_1\}$ joining p_k in H_k to a point p_1 in H_1 and which contains no other point in H_1. Similarly, if p' is any point in $\sum_{i=1}^{r} H_i$ distinct from p_k, p' is in some H_a for $a \leq r$, and there exists a path P' joining p' to a point p_1' in H_1 and containing no other point in H_1. Since H_1 is connected, there is a path P_1 in H_1 joining p_1 and p_1'. If these three paths are all disjoint, i.e., no two have any point in common except for the distinct points p_1 and p_1', they form, taken together, a path $\{p_k, \ldots, p_1, \ldots, p_1', \ldots, p'\}$ joining p_k and p'. If these paths are not disjoint except for distinct p_1 and p_1' there exists a point p_b in $P \cap H_b$ which is also a point of P'. If b is the greatest integer for which this is the case, then the segments $\{p_k, \ldots, p_b\}$ and $\{p', \ldots, p_b\}$, of P and p' respectively, when taken together, form a path $\{p_k, \ldots, p'\}$ joining p_k and p'.

To prove part 2, suppose that there is no non-null H_{q+1} such that $\{H_1, \ldots, H_q, H_{q+1}\}$ is an H-sequence. Then every point in $G(H_q)$ has a neighbor in one of the H_i for $1 \leq i < q$. Furthermore every point in H_q has a neighbor in H_{q-1}. From part 1, we know that $\sum_{i=1}^{q-1} H_i$ is connected and since it contains a neighbor of every point in H_q and of every point which has a neighbor in $\sum_{i=1}^{q} H_i$, it must also be 1-dense.

LEMMA 2.3. *If G'' is a connected subgraph and G' is a 1-dense subgraph containing G'', then there is a unique H-sequence based on G'' such that G' is equal to the union of all the subgraphs in the H-sequence.*

Define $H_1 \equiv G''$ and for $i>1$ let $H_i \equiv G' \cap G(H_{i-1}) - \sum_{j=1}^{i-2} G(H_j) - H_1$. Suppose that for some r, there are points of G' not in $\sum_{i \leq r} H_i$. At least one of these points, p say, must have a neighbor in some H_j for some $j \leq r$ since otherwise G' would not be connected. If $j<r$, p must be in H_i for $i \leq r$. This is contrary to hypothesis, and so p is in H_{r+1}. Thus since G' has only a finite number of points, there must be a positive integer q such that $G' \subset \sum_{i=1}^{q} H_i$,

[2] The notation $\Sigma_{A \in B}$ will denote the sum over all subgraphs A belonging to the set B of subgraphs, while $\Sigma_{C \subset A \in B}$ will denote the sum over all subgraphs A which belong to B and which contain the subgraph C, and $\Sigma_{C \supset A \in B}$ the sum over all A in B which are contained in C.

and it follows that $G'=\sum_{i=1}^{q}H_i$, since the H_i contain only points of G'. Furthermore the sequence $\{H_1, \ldots, H_q\}$, by its construction, is an H-sequence.

Suppose that there is a second H-sequence $\{H_1', \ldots, H_m'\}$ based on G'' such that $G'=\sum_{i=1}^{m}H_i'$, $H_i=H_i'$ for $i<j$ and $H_j\neq H_j'$. It then follows that H_j' must be unequal to $G'\cap G(H_{j-1})-\sum_{i=1}^{j-2}G(H_i)-H_1\equiv H_j$. We have $H_j'\subset H_j$, and therefore there are points in H_j-H_j' which are in graphs H_k' for $k>j$. Since these points have neighbors in H_{j-1}', this result is contrary to the definition of an H-sequence and thus $H_j=H_j'$. Since $H_1=H_1'=G''$, it follows by induction that the two H-sequences are identical.

An H-sequence based on a connected subgraph G'' with the property that the union of all its graphs is 1-dense will be called a D-minimal H-sequence if it contains no proper subsequence which is also an H-sequence based on G'' with this property. By 2.3, for a given connected subgraph G'' in G, to every 1-dense subgraph G' containing G'' there corresponds a unique H-sequence based on G'' such that G' is the union of the graphs in the H-sequence. This H-sequence contains a subsequence which is a D-minimal H-sequence based on G'' and which is also unique. We can see this by observing that given two H-sequences $\{H_i'\}$ and $\{H_i''\}$ which are subsequences of an H-sequence $\{H_i\}$, if for some r, $H_j'=H_j''=H_j$ for all $j\leq r$ and $H_{r+1}'\neq H_{r+1}''$, then either H_{r+1}' or H_{r+1}'' is $\neq H_{r+1}$ and one of these must be equal to H_j for $j>r+1$, which is impossible by definition of an H-sequence, Thus $\{H_i'\}$ and $\{H_i''\}$ must be identical if they are both D-minimal. Accordingly the H-sequences based on G'' and corresponding to 1-dense subgraphs containing G'' may be divided into families such that all the H-sequences in each family contain a particular D-minimal H-sequence based on G'', and then the H-sequence corresponding to a particular 1-dense subgraph containing G'' will belong to one, and only one, family. Let S be the set of all 1-dense subgraphs for which the corresponding H-sequences belong to a particular family; let $\{H_i, \ldots, H_q\}$ be the D-minimal H-sequence contained in every H-sequence of the family; and let $G_2=\Sigma_{i=1}^{q}H_i$. Then every H-sequence in the family may be denoted by $\{H_1, \ldots, H_q, H_{q+1}\}$ where H_{q+1} is the graph determined by any set of points, empty or nonempty, in $G(H_q)-\sum_{i=1}^{q-1}G(H_i)-H_1\equiv G_1$. There can be no H_k for $k>q+1$ in any H-sequence of the family, since otherwise H_k would have a neighbor in some H_i for $i\leq q$, which is impossible. By part 2 of 2.2, we know that G_1 is non-null if $q>1$. If $q=1$, G_1 is non-null if $G''=H_1$ is a proper subgraph. Assume, therefore that $G''\neq G$, so that $G_1\neq\phi$. The set S is the set of all 1-dense subgraphs in $G_1\cup G_2$ which contain G_2. By 2.1, $\sum_{G'\epsilon S}(-)^{\sigma(G')}=0$. and this must be true of every family of H-sequences based on G''. Thus if S_a is the set of all 1-dense subgraphs which contain G'', $\sum_{G'\epsilon S_a}(-)^{\sigma(G')}=0$. This establishes the theorem:

THEOREM 2.4. *If G'' is a proper connected subgraph in G and S_a is the set of all 1-dense subgraphs which contain G'', then $\sum_{G'\epsilon S_a}(-)^{\sigma(G')}=0$.*

The preliminary theorems and lemmas are now established which will make possible the determination of the value of f_k corresponding to any particular lattice point. Since the Moebius function is uniquely defined, any function must be equal to f_k if it satisfies the equations which define f_k recursively on the lattice of k-dense subgraphs. Consider in particular the function g_k whose domain is the set of subgraphs in S_k, defined by

$$g_k(G')=(-)^{n+\sigma(G')}$$

for $\phi\neq G'\epsilon S_1\cap S_k$,

$$g_k(G')=0$$

if G' is in $S_k-\phi$ but not 1-dense, and

$$g_k(\phi)=(-)^{n+1}\sum_{\phi\neq G'\epsilon S_1\cap S_k}(-)^{\sigma(G')}.$$

Since every subgraph which contains a 1-dense subgraph is 1-dense and thus k-dense, the set S of all subgraphs in S_k which contain a given 1-dense subgraph G'' in S_k is the set of all subgraphs which contain G''. Thus

$$-\sum_{G''<G'\epsilon S_k-G''}g_k(G')=(-)^{n+1}\left[\sum_{G'\epsilon S}(-)^{\sigma(G')}-(-)^{\sigma(G'')}\right]$$
$$=(-)^{n+\sigma(G'')}=g_k(G'').$$

which follows by 2.1 if we set $G_1=G$ and $G_2=G''$. If $G''\epsilon S_k-S_1$, the set of all subgraphs which properly contain G'' and for which g_k does not vanish is identical with the set of S of all 1-dense subgraphs containing G''. Since G'' is not 1-dense, $G''\neq G$, and since G'' is k-dense, it is connected. Therefore, by 2.4,

$$-\sum_{G''\subset G'\epsilon S_k-G''}g_k(G')=(-)^{n+1}\sum_{G'\epsilon S}(-)^{\sigma(G')}=0=g_k(G'').$$

Thus it is established that $g_k(G')$ satisfies the recurrence relations which define f_k for all $G'\neq\phi$ in S_k, and so $g_k(G')=f_k(G')$ for $G'\neq\phi$. It then follows from its definition that $g_k(\phi)=f_k(\phi)$. Accordingly, we have proved

THEOREM 2.5. *On any lattice of k-dense subgraphs, $g_k=f_k$.*

Since for any non-null G' in S_k, $f_k(G')$ depends only on $\sigma(G')$ and on whether or not $G'\epsilon S_1$, attention will be given to $f_k(\phi)$, and in particular $f_1(\phi)$, for which the value depends on whether G is completely connected, or contains an articulator or isthmus, and on other properties of G.

THEOREM 2.6. *If G is completely connected, $f_1(\phi)=(-)^n$.*

If G is completely connected, every point is a neighbor of all other points and is therefore 1-dense, as, in fact, is every non-null subgraph. Thus, for every $k\geq 1$, G contains $\binom{n}{k}$ 1-dense subgraphs hav-

ing k points each, so that by 2.5,

$$f_1(\phi)=(-)^{n+1}\sum_{k=1}^{n}\binom{n}{k}(-)^k=(-)^n,$$

a result which follows from the binomial expansion of $(1-1)^n$.

LEMMA 2.7. *If S is a set of subgraphs of G, F_i for $i=1,2,\ldots,n$ the set of all families of subgraphs in S such that each family contains precisely i subgraphs, for any particular family C, $B(C)$ is the union of all the subgraphs in C, and S' is the set of all subgraphs of G which belong to a given set R and which contain one of the subgraphs in S, then*

$$\sum_{G'\epsilon S'}(-)^{\sigma(G')}=-\sum_{i=1}^{n}\sum_{C\epsilon F_i}\sum_{B(C)\in G''\epsilon R}(-)^{\sigma(G'')+i}. \quad (A)$$

Consider a particular subgraph $G'\epsilon S'$ and suppose that G' contains precisely q subgraphs from S which form a family F_q. The subgraph G' will occur once in the triple sum in (A) for every subfamily of the family F_q, and since F_q contains $\binom{q}{k}$ subfamilies of precisely k graphs each, the total contribution to the triple sum in (A) of the summands corresponding to G' is

$$-\sum_{k=1}^{q}\binom{q}{k}(-)^{k+\sigma(G')}=(-)^{\sigma(G')}.$$

Since this result holds for every subgraph G' in S', the right and left members of (A) are equal.

THEOREM 2.8. *If G contains at least k disjoint subgraphs, each of which disconnects G and at least one of which is an isthmus, $f_k(\phi)=0$.*

If $G_1,\ldots,G_k$ are k disjoint subgraphs each of which disconnects G, then every 1-dense subgraph contains at least k points, one from each G_i $(i=1,\ldots,k)$. Otherwise for some j, $(1\leq j\leq k)$, $G-G_j$ would be 1-dense and thus connected. Thus $S_1=S_1\cap S_k$ and $f_k(\phi)=g_k(\phi)=g_1(\phi)$ by 2.5. Suppose one of the G_i, G_1 say, is completely connected, so that every subgraph of G_1 is a connected proper subgraph of G. If S is the set of non-null subgraphs of G_1 and S' the set of 1-dense subgraphs containing a graph from the set S, we have $S'=S_1$, and thus

$$g_1(\phi)=(-)^{n+1}\sum_{G\epsilon S'}(-)^{\sigma(G')}.$$

If C is any subfamily of S and $B(C)$ the union of the graphs in C, $B(C)$ is connected because G_1 is completely connected, and thus by 2.4,

$$\sum_{B(C)\in G''\epsilon S_1}(-)^{\sigma(G'')}=0.$$

Since this is true of every family C, we find from 2.7 that $g_1(\phi)=0$.

Let us now suppose that S denotes the set of minimal subgraphs which belong to $S_1\cap S_k$, i.e., the subgraphs with this property which have no proper subgraphs belonging to $S_1\cap S_k$, while R is the set of 1-dense subgraphs. Since every 1-dense subgraph having at least k points contains a minimal subgraph with this property, 2.5 implies if S' is the set of all subgraphs in R which contain one of the subgraphs in S, then

$$f_k(\phi)=(-)^{n+1}\sum_{G'\epsilon S'}(-)^{\sigma(G')}.$$

If C is any family of subgraphs in S and $B(C)$ is the union of the graphs in C, then $B(C)$ is connected, since the union of two 1-dense subgraphs is 1-dense [2, lemma 2.1] and therefore connected. If G is not equal to the union of all the subgraphs in S, $B(C)$ is a proper connected subgraph. Then by 2.4,

$$\sum_{B(C)\in G''\epsilon R}(-)^{\sigma(G')}=0.$$

Since this holds for every family C, 2.7 implies

$$\sum_{G'\epsilon S'}(-)^{\sigma(G'')}=0,$$

so that f_k vanishes. Thus we have proved

THEOREM 2.9. *If G is not equal to the union of all the minimal subgraphs with the property that they are 1-dense and contain at least k points, then $f_k(\phi)=0$.*

From 2.9 is is seen that $f_1(\phi)=0$ unless G is equal to the union of the D_1-minimal subgraphs. If $G=\Gamma_1$ and that D_1-minimal subgraphs are mutually disjoint, it has been proved elsewhere [3, theorem 2.6] that G is completely connected, so that, by 2.6, $f_1(\phi)=(-)^n$.
Thus we have proved

THEOREM 2.10. *If the D_1-minimal subgraphs are mutually disjoint, $|f_1(\phi)|\leq 1$.*

It has been proved [3, theorem 2.4] that if $n>1$, G contains at least two D_1-maximal subgraphs. Suppose $n>1$ and there is a D_1-minimal subgraph G' which is contained in every D_1-maximal sugraph, and G'' is D_1-minimal, $G''\neq G'$. Then there is a point p in $G'-G''$, and $G''+(G-p)$ is D_1-maximal. This impossible, since this D_1-maximal subgraph does not contain G'. Thus $\Gamma_1=G'$, and Γ_1 is a proper subgraph, so that by 2.9, $f_1(\phi)=0$. This result is Hall's theorem [4] for the special case of the lattice of 1-dense subgraphs. It may be stated in the form:

THEOREM 2.11. *On the lattice of 1-dense subgraphs of G, if $n>1$ and ϕ is not the g.l.b. of any set of D_1-maximal subgraphs, then $f_1(\phi)=0$.*

It has been previously proved [3] that either G is completely connected, in which case $|f_1(\phi)|=1$ by 2.6, or G contains a disconnecting subgraph which in turn must contain an articulator or an isthmus. If G contains an isthmus, $f_1(\phi)=0$ by 2.7, and thus we have proved

THEOREM 2.12. $|f_1(\phi)|\leq 1$ *unless G contains an articulator.*

3. Graphs Containing Articulators

A simple example of a graph containing an articulator is an r-cycle for $r>3$. Since each point is connected to only two other points, the two neighbors of any given point in the r-cycle constitute a disconnecting subgraph G', which is not connected, each point of which is connected to both graphs in the partition of $G-G'$, and which is therefore a $[2,1]$-articulator.

THEOREM 3.1. *If G contains an n-cycle, then $f_k(\phi)=-1$ for $k\neq n-1$, and $f_{n-1}(\phi)=n-1$.*

If G contains an n-cycle, every point is one of a sequence $\{p_1, \ldots, p_n\}$ such that each point is a neighbor only of those which immediately precede and immediately follow in the sequence, except for p_1 and p_n which are neighbors. We have from 2.5, $f_n(\phi)=(-)^{n+1}\cdot(-)^n=-1$. For any i, $G-p_i$ is 1-dense since $P\equiv\{p_{i+1}, \ldots, p_n, p_1, \ldots, p_{i-1}\}$ is a path connecting p_{i+1} and p_{i-1}, p_{i-1} being taken to mean p_n if $i=1$, which are neighbors of p_i, and any two other points in $G-p_i$ are joined by a subpath contained in P. Similarly for any i, $G-p_i-p_{i-1}$ is connected since there exists a path $\{p_{i+1}, \ldots, p_{i-2}\}$ containing all the other points. On the other hand, if p_a and p_b are not neighbors and $a<b$, there exists p_c and p_d such that $a<c<b$ and either $d>b$ or $d<a$. It is easily shown by induction that any path joining p_c and p_d must contain p_a or p_b, and thus $G-p_a-p_b$ is not connected. Thus for $n\geq 3$, $S_1-\phi$ consists of G plus n subgraphs $G-p_i$ $(i=1, \ldots, n)$, and n subgraphs $G-p_i-p_{i-1}$, so that by 2.5 $f_1(\phi)=(-)^{n+1}\,[n(-)^{n-2}+n(-)^{n-1}+(-)^n]=-1$. Also for $n\geq 2$ we have $f_{n-1}(\phi)=(-)^{n+1}[n(-)^{n-1}+(-)^n]=n-1$.

All graphs considered hitherto have been such that $|f_1(\phi)|\leq 1$. Consideration of graphs containing articulators, however, will show that there exist graphs for which $f_1(\phi)$ assumes arbitrarily large positive or negative values. First we must prove

LEMMA 3.2 *If G contains an articulator G' such that each point in $G-G'$ is a neighbor of every point of G' then*

$$f_1(\phi)=(-)^{n+1}[1+\sum_{G'\supset G''\in S_1-\phi}(-)^{\sigma(G'')}].$$

If H is any subgraph containing a point p' in G' and a point p in $G-G'$, then p and p' are neighbors; every point in G' is a neighbor of p; and every point in $G-G'$ is a neighbor of p'. Thus the subgraph $p+p'$ is 1-dense as is H which contains it. If $\sigma(G')=m$, there are $\binom{m}{k}$ subgraphs contained in G' having k points each and $\binom{n-m}{r}$ subgraphs contained in $G-G'$ having r points each which can be combined to give $\binom{m}{k}\cdot\binom{n-m}{r}$ 1-dense subgraphs, each with k points in G' and r points in $G-G'$. If S_1' is the set of all graphs in S_1 which have points in both G' and $G-G'$, we have

$$\sum_{H\in S_1'}(-)^{\sigma(H)}=\Big[\sum_{k=1}^{m}\binom{m}{k}(-)^k\Big]\Big[\sum_{r=1}^{n-m}\binom{n-m}{r}(-)^r\Big]=1.$$

Since there are no 1-dense subgraphs contained in $G-G'$, which would otherwise be connected, every graph in S_1-S_1' is contained in G'. Thus, by 2.5,

$$f_1(\phi)=(-)^{n+1}\Big[\sum_{H\in S_1'}(-)^{\sigma(H)}+\sum_{G'\supset G''\in S_1-\phi}(-)^{\sigma(G'')}\Big]$$

$$=(-)^{n+1}\Big[1+\sum_{G'\supset G''\in S_1-\phi}(-)^{\sigma(G'')}\Big].$$

THEOREM 3.3. *If q is any positive integer, there exist graphs G and G' for which $f_1(\phi)=q$ and $f_1(\phi)=-q$ respectively.*

Suppose there exists a graph G_a for which $f_1(\phi)=r$. Consider the graph G_b which has the following properties: G_b consists of an articulator G_b' isomorphic with G_a plus s points, $p_1, \ldots, p_s$ at least two of which are not neighbors, and each of which is a neighbor of every point of G_b'. By 3.2, we have for G_b the result that

$$f_1(\phi)=(-)^{\sigma(G_a)+s+1}\Big[1+\sum_{G_b'\supset G''\in S_1-\phi}(-)^{\sigma(G'')}\Big].$$

Since any subgraph which is 1-dense in G_b' contains neighbors of $p_1, \ldots, p_s$ and is therefore 1-dense in G_b, it follows that

$$f_1^b(\phi)\equiv(-)^{\sigma(G_a)+1}\sum_{G_b'\supset G''\in S_1-\phi}(-)^{\sigma(G'')}$$

is the function $f_1(\phi)$ defined on the lattice of 1-dense subgraphs of G_b'. Furthermore, if two graphs are isomorphic, to every 1-dense subgraph of one there corresponds a unique 1-dense subgraph of the other, and thus $f_1(\phi)$ for G_b' is equal to $f_1(\phi)$ for G_a, which we shall denote by $f_1^a(\phi)$. Thus

$$f_1(\phi)=(-)^{\sigma(G_a)+s+1}+(-)^s f_1^a(\phi).$$

If $\sigma(G_a)$ is even, $f_1(\phi)=(-)^s[r-1]$. In particular we can suppose G_a contains an n-cycle, n even and >2, so that $r=-1$ and let s be odd, so that $\sigma(G_b)$ is odd and $f_1(\phi)=2$. If s were even, then $f_1(\phi)=-2$. However, we have also proved that if there exists G_a such that $\sigma(G_a)$ is odd and $f_1^a(\phi)=r$, then there exists a graph G_a' for which $f_1(\phi)=(-)^s[r+1]$, containing $\sigma(G_a)+s$ points. If s is even, $\sigma(G_a')$ is odd and for G_a', $f_1(\phi)=r+1$. If s is odd, $f_1(\phi)$ for G_a' is $-[r+1]$. By induction, it follows that for any integer q such that $q=-1$ or $|q|\geq 2$, there exists a graph G for which $f_1(\phi)=q$. If G is completely connected and $n=2$, $f_1(\phi)=1$ by 2.6, and thus the theorem is proved for all positive integers q.

THEOREM 3.4. *If p and p' are two distinct points of G, and S is the set of all families of paths joining p and p' such that each family contains n points among them, then the families of S, together with the null family, ϕ_S, form a lattice under the relation of set inclusion.*

If a family F of paths contains all points of G, any family containing F is also in S, and thus the l.u.b. of two families in S is their union. If the intersection of two families in S is not in S, it cannot contain any family of S. Thus the g.l.b. of two families is either their intersection or ϕ_S. For any pair of distinct points p and p', the lattice formed by the families of paths which join p and p', such that the paths of each family contain all points of G, will be called a *lattice of path sets* associated with G and joining p and p'. For any given lattice formed by a set of S of path sets joining a given pair of points, $\tau(F)$ will denote the number of distinct paths in each lattice element $F \neq \phi_S$, two paths being called "distinct" if they differ in at least one point, and m_S will denote the greatest positive integer with the property that for some F, $m_S = \tau(F)$. The symbol h_S will denote the Moebius function on the lattice formed by path sets from S.

THEOREM 3.5. *If S is a set of families of paths, including the null family, which form a lattice of path sets associated with G, then for each $F \epsilon S$,*

$$h_S(F) = (-)^{m_S + \tau(F)}$$

for $F \neq \phi_S$, and if $m_S \geq 1$,

$$h_S(\phi_S) = (-)^{m_S+1} \sum_{\phi_S \neq F \epsilon S} (-)^{\tau(F)}.$$

The proof is very similar to that of 2.5 and will be left to the reader.

If G contains a $[2, k]$-articulator consisting of points p and p', every 1-dense subgraph must contain either p or p' since otherwise $G - p - p'$ would be 1-dense and therefore connected. Accordingly if R is the set of 1-dense subgraphs in G and S' the set of all subgraphs in R which contain p, p', or $p+p'$, we have $R = S'$, and then 2.5 and 2.7 imply that

$$f_1(\phi) = (-)^{n+1} \sum_{G' \epsilon S'} (-)^{\sigma(G')} = (-)^{n+1} \times \Big[\sum_{p \subset G'' \epsilon R} (-)^{\sigma(G'')} + \sum_{p' \supseteq G'' \epsilon R} (-)^{\sigma(G'')} - \sum_{p+p' \subset G'' \epsilon R} (-)^{\sigma(G'')} \Big].$$

The first two terms in the brackets vanish by 2.4 since p and p' are each connected proper subgraphs. Every 1-dense subgraph containing both p and p' must contain a path from the set S of all paths joining these two points, and so again employing 2.7, we find that

$$f_1(\phi) = (-)^{n+1} \sum_{i=1}^{n} \sum_{C \epsilon F_i} \sum_{B(C) \subset G'' \epsilon R} (-)^{\sigma(G'')+i},$$

where F_i is the set of all i-ples of distinct paths from S, and C, $B(C)$, are defined as in the hypothesis of 2.7. For each family C, $B(C)$ is connected since the union of two connected subgraphs having a point in common is connected. Accordingly, by 2.4, $\sum_{B(C) \subset G'' \epsilon R} (-)^{\sigma(G'')} = 0$ unless $B(C) = G$, i.e., unless C is an element of the lattice of path sets joining p and p', in which case $\sum_{B(C) \subset G'' \epsilon R} (-)^{\sigma(G'')+i} = (-)^{n+\tau(C)}$. If S'' is the set of lattice elements, then $f_1(\phi) = -\sum_{\phi_{S''} \neq F \epsilon S''} (-)^{\tau(F)}$, provided $\phi_{S''}$ is not the only element in S'', and $f_1(\phi) = 0$ otherwise. This result implies

THEOREM 3.6. *If G contains a $[2, k]$-articulator and S is the set of elements forming the lattice of path sets associated with G and joining the points of the articulator, then $f_1(\phi) = (-)^{m_S} h_S(\phi_S)$ if S contains a nonempty set of paths, and $f_1(\phi) = 0$ otherwise.*

Theorem 3.6 can be used to show that if G contains a $[2,k]$-articulator consisting of points p and p' and $f_1(\phi) \neq 0$, then $f_1(\phi)$ can be written as a product of factors, one factor for each connected graph in the partition of $G-G'$. Any path connecting p and p' cannot contain points from more than one graph in the partition of $G-G'$, and so any set of paths forming one of the set S of elements of the lattice of path sets joining p and p' must contain a subset contained in $G_i + p + p'$ and containing all the points of G_i, for each connected graph G_i ($i = 1, \ldots, w \geq k+1$) in the partition of $G-G'$. Thus if S_i ($i = 1, \ldots, w$) denotes the elements of the lattice of path sets associated with $G_i + p + p'$ and joining p and p', we have by 3.5,

$$h_S(\phi_S) = (-)^{m_S+1} \prod_{i=1}^{w} \Big(\sum_{\phi_{S_i} \neq G'' \epsilon S_i} (-)^{\tau(G'')} \Big) = (-)^{m_S+1} \prod_{i=1}^{w} \{ (-)^{m_{S_i}+1} h_{S_i}(\phi_{S_i}) \}.$$

Since $m_S = \sum_i m_{S_i}$, we have established

THEOREM 3.7. *If G contains a $[2, k]$-articulator G', G_i for $i = 1, \ldots, k+1$ are the connected graphs in the partition of $G-G'$, and for each G_i, S_i is the set of elements which form the lattice of path sets associated with $G_i \cup G'$ and joining the points of the articulator, then $f_1(\phi) \neq 0$ implies*

$$f_1(\phi) = (-)^{m_S+k} \prod_{i=1}^{k+1} h_{S_i}(\phi_{S_i}).$$

4. Associated Graphs

Given a graph G and a subgraph G' which is not properly contained in a completely connected subgraph, the *set of graphs associated with G and G'* is defined to be set of all graphs G'' which have the following properties: (1) A one-one correspondence exists between the points of G and those of G'' such

that neighbors in G are mapped into neighbors in G''; (2) G'' contains a maximal completely connected subgraph, which will be called $K[G'']$, with the property that $K[G'']$ contains the points in G'' which correspond to G' in G; (3) any pair of neighbors in G'' such that both are not in $K[G'']$, is mapped into a pair of neighbors in G. If G_1 and G_2 are any two graphs in the set associated with G and G', we shall say $G_1 \geq G_2$ whenever the points in G corresponding to $K[G_2]$ are a subset of the points corresponding to $K[G_1]$, and $G_1 > G_2$ whenever $G_1 \geq G_2$ but $G_1 \neq G_2$.

THEOREM 4.1. *Given G and a subgraph G', the set of graphs associated with G and G', if nonempty, form a lattice under the relation $\geq$.*

The proof will be left to the reader. We shall denote by $h(G''; G, G')$ the Moebius function evaluated for any G'' in the lattice of graphs associated with G and G', while $f_1(\phi; G'')$ will denote the Moebius function evaluated for ϕ on the lattice of subgraphs 1-dense in G''. The function h_1 whose domain is the set of graphs associated with G and G' is defined by $h_1(G''; G, G') \equiv (-)^{\sigma(K[G'']) + n}$, where G'' is any graph of the set.

THEOREM 4.2. *On the lattice formed by the set of graphs associated with G and G', $h_1 = h$.*

If G_m is the greatest element of the lattice $K[G_m] = G_m$ and $\sigma(K[G_m]) = n$, so that $h_1(G_m; G, G') = 1$. If $G'' < G_m$ and $w \equiv \sigma(K[G_m]) - \sigma(K[G''])$ there are $\binom{w}{k}$ graphs H which satisfy the condition that $G'' < H \leq G_m$ and $\sigma(K[H]) - \sigma(K[G'']) = k$. Therefore

$$-\sum_{H > G''} h_1(H; G, G') = (-)^{\sigma(K[G'']) + n + 1} \times \sum_{k=1}^{w} \binom{w}{k} (-)^k = (-)^{\sigma(K[G'']) + n} = h_1(G''; G, G').$$

Since h_1 satisfies the recurrence relations which uniquely determine h, the theorem follows.

Let $R[G'']$ be the set of subgraphs 1-dense in G'' and $Q[G'']$ the set of all subgraphs in $R[G'']$ which contain at least one point from $K[G'']$.

LEMMA 4.3. *If G'' is any graph in the set associated with G and G' such that $K[G''] \neq G''$,* then

$$f_1(\phi; G'') = (-)^{\sigma(G'') + 1} \sum_{H \epsilon R[G''] - Q[G'']} (-)^{\sigma(H)}.$$

by 2.5,

$$f_1(\phi; G'') = (-)^{\sigma(G'') + 1} \left[\sum_{H \epsilon Q[G'']} (-)^{\sigma(H)} + \sum_{H \epsilon R[G''] - Q[G'']} (-)^{\sigma(H)} \right].$$

By 2.7, the first sum in the bracket can be expressed as a sum over families of subsets of $K[G'']$. If C is such a family and $B(C)$ the union of the graphs in the family, $B(C)$ is a connected proper subgraph because $K[G'']$ is completely connected and $\neq G''$, and thus

$$\sum_{B(C) \supseteq H \epsilon K[G'']} (-)^{\sigma(H)} = 0$$

by 2.4. Since this is true for every family C of subgraphs of $K[G'']$, we have

$$\sum_{H \epsilon Q[G'']} (-)^{\sigma(H)} = 0.$$

THEOREM 4.4. *If G'' is a graph belonging to the set associated with G and G', then*

1) *If $G'' - K[G'']$ is 1-dense in G'',*

$$\sum_{H \geq G''} h(H; G, G') f_1(\phi; H) = 0.$$

2) *If $G'' - K[G'']$ is not 1-dense in G'',*

$$\sum_{H \geq G''} h(H; G, G') f_1(\phi; H) = (-)^{\sigma(G'')}.$$

By 4.2 and 4.3, we have for $G'' \neq K[G'']$

$$-\sum_{H > G''} h(H; G, G'') f_1(\phi, H) = \sum_{H > G''} \sum_{H' \epsilon R[H] - Q[H]} (-)^{\sigma(K[H]) + \sigma(H')} + (-)^{n+1}. \quad \text{(B)}$$

If H' is any subgraph of G'' which is one of the set S associated with G and G', H' corresponds to a unique subgraph $H_1' \subset G$. In turn H_1' corresponds to a unique subgraph H'' in any other graph of S, and these two correspondences determine a correspondence between H' and H''. If $H > G''$, every subgraph of $H - K[H]$ corresponds to a subgraph in $G'' - K[G'']$. If $G'' - K[G'']$ is not 1-dense, it contains no 1-dense subgraph, and so there can be no corresponding 1-dense subgraph in $H - K[H]$ and $R[H] - Q[H]$ is empty for $H \geq G''$. Therefore, if $G'' \neq K[G'']$, $-h(G''; G, G')\, f_1(\phi; G'')$ can be added to both members of (B), and then the double sum vanishes, proving part 2 for this case. If $G'' = K[G'']$, $f_1(\phi; G'') = (-)^n$ and part 2 is obvious.

Suppose $G'' - K[G'']$ is 1-dense, and $H' \epsilon R[G''] - Q[G'']$. If $w \equiv \sigma(G'' - H' - K[G''])$ and $k = 1, \ldots, w$, there are $\binom{w}{k}$ graphs $H > G''$ with the property that $K[H] = K[G''] + k$ and $H - K[H]$ contains a subgraph which corresponds to H'. The contribution to the double sum in (B) of all the graphs which correspond to a particular $H' \epsilon R[G''] - Q[G'']$ is

$$\sum_{k=1}^{w} \binom{w}{k} (-)^{\sigma(K[G'']) + k + \sigma(H')} = (-)^{\sigma(K[G'']) + 1 + \sigma(H')}.$$

(B) contains a similar contribution for every $H' \epsilon R[G''] - Q[G'']$ except $G'' - K[G'']$. Thus

$$-\sum_{H > G''} h(H; G, G') f_1(\phi; H) = h(G''; G, G') f_1(\phi; G'') - (-)^{\sigma(K[G'']) + 1 + \sigma(G'' - K[G''])} + (-)^{n+1}.$$

Since $\sigma(G'') = \sigma(G) = n$, part 1 follows.

5. References

[1] R. E. Nettleton, Some generalized theorems on connectivity (to be published).

[2] G. Birkhoff, Lattice theory, Am. Math. Soc. Colloq. Pub. **25** (1948).

[3] R. E. Nettleton, K. Goldberg, and M. S. Green, Dense subgraphs and connectivity, Can. J. Math **11**, 262 (1959).

[4] P. Hall, Quar. J. **7**, 134 (1936).

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The Minimum of a Certain Linear Form[1]

Karl Goldberg

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The positive minimum of the integral linear form $L(x_1, \ldots, x_n)=a_1x_1+\ldots+a_nx_n$ is found subject to the conditions $a_i>0$ and $L(x_1, \ldots, x_n)\geq 2a_ix_i$ for $i=1,2, \ldots, n$.

Let $a_1\leq a_2\leq \ldots \leq a_n$ be $n\geq 3$ positive integers. We seek the positive minimum M of the linear form

$$L(x_1,x_2, \ldots ,x_n)=a_1x_1+a_2x_2+\ldots+a_nx_n$$

over all non-negative integers $x_1, x_2, \ldots, x_n$ such that

$$L(x_1,x_2, \ldots, x_n)\geq 2a_ix_i \tag{1}$$

for all $i=1, 2, \ldots, n$.

Let $[a_1,a_2]$ denote the least common multiple of a_1 and a_2.

For each $i=3, 4, \ldots, n$, define r_i in the following way: If either a_1 or a_2 divides a_i, or if $a_i=a_j$ for some $j\neq i$, set $r_i=0$. Otherwise, let r_i be the minimum of the least non-negative residues modulo a_1 of

$$a_2-a_i,\ 2a_2-a_i, \ldots, [(a_i-1)/a_2]a_2-a_i.$$

We shall prove

THEOREM: *M is the minimum of* $2[a_1,a_2]$, $2a_3+r_3$, $2a_4+r_4, \ldots, 2a_n+r_n$.

As a consequence we have the inequality

$$2a_3+a_1-1\geq M\geq 2a_2.$$

Also, if $L(x_1,x_2, \ldots, x_n)=M$, then at most three of the x_k are positive. At least two must be positive. If exactly two are positive, then either $x_1=[a_1,a_2]/a_1$ and $x_2=[a_1,a_2]/a_2$, or $x_1=a_i/a_1$ and $x_i=1$, or $x_2=a_i/a_2$ and $x_i=1$, or $x_i=x_j=1$ for some $j>i\geq 3$. If three of the x_k are positive, then both x_1 and x_2 are positive; the other positive x_i equals 1 and we have $x_1=-[(a_2x_2-a_i)/a_1]$ for that i. Under any conditions M is achieved only with $x_i\leq 1$ for all $i\geq 3$. We shall prove all this.

M. Newman [2] refers to our theorem in the case $n=3$. We shall treat this case first.

We have $a_1\leq a_2\leq a_3$, and we want to find the positive minimum M of the linear form

$$L(x_1,x_2,x_3)=a_1x_1+a_2x_2+a_3x_3$$

over all non-negative integers x_1, x_2, x_3 satisfying

$$L(x_1,x_2,x_3)\geq 2a_ix_i \qquad i=1,2,3. \tag{2}$$

Let $x_1'=-[(a_2-a_3)/a_1]$. Because $a_2\leq a_3$, x_1' is non-negative. It satisfies

$$a_1-1\geq a_2-a_3+a_1x_1'\geq 0. \tag{3}$$

Because $a_1\leq a_2$, this implies

$$a_3\geq a_3-(a_2-a_1)-1\geq a_1x_1'. \tag{4}$$

Now consider $L(x_1',1,1)=a_1x_1'+a_2+a_3$. From (3) we have

$$2a_3+a_1-1\geq L(x_1',1,1)\geq 2a_3. \tag{5}$$

We know that $2a_3\geq 2a_2$, so that $L(x_1',1,1)\geq 2a_3\geq 2a_2$. Finally, (4) yields $L(x_1',1,1)\geq 2a_3\geq 2a_1x_1'$.

This proves that $x_1=x_1'$, $x_2=x_3=1$ satisfies (2). It follows that the left-hand inequality in (5) holds for M:

$$2a_3+a_1-1\geq M. \tag{6}$$

From this point we assume that x_1, x_2, x_3 satisfy (2) and

$$L(x_1,x_2,x_3)=M.$$

Since $L(x_1,x_2,x_3)\geq 2a_3x_3$, we have from (6) and $a_3\geq a_1$ that $x_3=0$ or $x_3=1$.

If $x_3=0$, than (2) implies $a_1x_1=a_2x_2$. Under this condition the minimum value of $L(x_1,x_2,x_3)$ is $2[a_1,a_2]$, occurring for $x_1=[a_1,a_2]/a_1$ and $x_2=[a_1,a_2]/a_2$.

From now on $x_3=1$. From $M=a_1x_1+a_2x_2+a_3$ and (2), we have $M\geq 2a_3$. From (6) we have

$$2a_3+a_1-1\geq 2a_3+a_1x_1+(a_2x_2-a_3),$$

from which it follows that $x_1>0$ implies $a_3-1\geq a_2x_2$. If $x_1=0$, then (2) implies $a_2x_2=a_3x_3$, so that $M=2[a_2,a_3]$.

[1] The preparation of this paper was supported in part by the Office of Naval Research.

[2] M. Newman, Construction and application of a class of modular functions, II. Proc. London Math. Soc. **9**, 373 (1959).

But $2[a_2,a_3]>2a_3+a_1-1$ unless a_2 divides a_3. Thus $x_1=0$ is possible only if a_2 divides a_3, in which case $x_2=a_3/a_2$ and $M=2a_3$. Similarly $x_2=0$ is possible only if a_1 divides a_3, in which case $x_1=a_3/a_1$ and $M=2a_3$. Since a_3 divisible by either a_1 or a_2 leads to $M=2a_3$ which is the best possible result with $x_3=1$, we may now assume that neither a_1 nor a_2 divides a_3 and that $x_1x_2>0$.

With $x_1>0$ we must have $a_3-1\geq a_2x_2$. Fix x_2. We shall find that permissible value of x_1 which minimizes $L(x_1,x_2,x_3)=a_1x_1+a_2x_2+a_3$. Clearly this is the least positive value of x_1 satisfying (2). We have

$$L(x_1,x_2,x_3)=a_1x_1+1+(a_3-1-a_2x_2)+2a_2x_2>2a_2x_2$$

for any value of x_1. The other inequalities require

$$\frac{a_3+a_2x_2}{a_1}\geq x_1\geq \frac{a_3-a_2x_2}{a_1}.$$

Since $2a_2x_2\geq 2a_2>a_1$, there are values of x_1 satisfying these inequalities. The least such x_1 is the least integer greater than or equal to $(a_3-a_2x_2)/a_1$. This last quantity is positive, so this value of x_1 is positive. It can be written

$$x_1''=-\left[\frac{a_2x_2-a_3}{a_1}\right].$$

Let $r_3(x_2)$ be the least non-negative residue modulo a_1 of $a_2x_2-a_3$. Then $r_3(x_2)=a_2x_2-a_3+a_1x_1''$. It follows that

$$L(x_1'',x_2,1)=2a_3+r_3(x_2).$$

We want the least of these values for x_2 lying between 1 and $[(a_3-1)/a_2]$. Under our assumptions on the divisibility of a_3, this is just $2a_3+r_3$ with r_3 as defined in the theorem. This proves the theorem for $n=3$.

Now assume $n>3$. We have $a_1\leq a_2\leq \ . \ . \ . \ \leq a_n$, and we want to find the positive minimum M of the linear form $L(x_1,x_2, \ . \ . \ ., x_n)$ over all non-negative integers $x_1,x_2, \ . \ . \ ., x_n$ satisfying (1).

If x_1,x_2,x_3 satisfy (2), then $x_1,x_2,x_3, 0, \ . \ . \ ., 0$ satisfy (1). Therefore our new M satisfies (6). Let

$$L(x_1,x_2, \ . \ . \ ., x_n)=M.$$

Then

$$2a_3+a_1-1\geq a_1x_1+a_2x_2+a_3x_3+ \ . \ . \ . \ +a_nx_n$$
$$\geq a_1(x_1+x_2)+a_3(x_3+ \ . \ . \ . \ +x_n).$$

It follows that

$$x_3+x_4+ \ . \ . \ . \ +x_n\leq 2$$

and that $x_3+x_4+ \ . \ . \ . \ +x_n=2$ requires $x_1+x_2=0$. On the other hand,

$$3a_i>2a_3+a_1-1\geq L(x_1,x_2, \ . \ . \ ., x_n)\geq 2a_ix_i,$$
$$i=3,4, \ . \ . \ ., n,$$

implies

$$x_i\leq 1, \qquad i=3,4, \ . \ . \ ., n.$$

Assume $x_3+x_4+ \ . \ . \ . \ +x_n=2$. Then $x_i=x_j=1$ for some $i,j\geq 3$ and all other $x_k=0$. Then (1) implies $a_i=a_j$ and $M=2a_i$. Again this is the best possible result with $x_i=1$.

If $x_3+x_4+ \ . \ . \ . \ +x_n=0$, then (1) implies $a_1x_1=a_2x_2$. As before, this implies $M=2[a_1,a_2]$.

If $x_3+x_4+ \ . \ . \ . \ +x_n=1$, then $x_i=1$ for some $i\geq 3$ and all other $x_k=0$ for $k\geq 3$. The problem then reverts to the case $n=3$ with a_i replacing a_3. Our previous arguments complete the proof of the theorem, and the statements in the subsequent paragraph.

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Space of k-Commutative Matrices[1]

Marvin Marcus[2] and N. A. Khan[3]

(August 14, 1959)

Let $[A, X]=AX-XA$ and $[A, X]_k=[A, [A, X]_{k-1}]$. Those matrices X which "k-commute" with a fixed matrix A are investigated. In particular, the dimension of the null space of the linear transformation $T(X)=[A, X]_k$ when A is nonderogatory is determined.

1. Introduction

Let A be a fixed N-square complex matrix and let $[A, X]=AX-XA$, $[A, X]_k=[A, [A, X]_{k-1}]$. It is easily checked that

$$[A,X]_k=\sum_{s=0}^{k}(-1)^s\binom{k}{s}A^{k-s}XA^s. \tag{1.1}$$

It is clear that the set of X such that $[A, X]_k=0$ is a linear subspace of the space M_N of all N-square complex matrices. This subspace is denoted by $\mathfrak{A}_k(A)$. In theorem 1 is determined the dimension of $\mathfrak{A}_k(A)$ in terms of the degrees of the elementary divisors of A when there is exactly one elementary divisor for each eigenvalue. Let E_q denote the set of matrices in M_N with precisely q distinct eigenvalues. In theorem 2 it is shown that in case $k\geq 2(N-q)+1$, then

$$\min_{A\in E_q\cap M_N}\dim \mathfrak{A}_k(A)=R(Q+1)^2+(q-R)Q^2$$

where $N=Qq+R$, $0\leq R<q$. The maximum is also found.

2. Results

Let T denote the linear transformation on M_N defined by $T(X)=[A, X]$ and we note that $T^k(X)=[A, X]_k$. With respect to the basis E_{ij} in M_N, ordered lexicographically, we check that T has the matrix representation $I_N\otimes A-A'\otimes I_N$. The notation is the following: E_{ij} is the N-square matrix with 1 in position i, j, 0 elsewhere; $I_N\otimes A$ denotes the Kronecker product of the N-square identity matrix I_N with A. It is clear that one may assume A is in Jordan canonical form

$$A=\sum_{s=1}^{q}{}^{\bullet}\,J_s, \tag{2.1}$$

where $\Sigma^{\bullet}$ indicates direct sum and the J_s are the Jordan blocks corresponding to the distinct eigenvalues λ_s, $s=1, \ldots, q$, of A. If $T_k(X)=0$ and X is partitioned conformally with the partitioning (2.1) of A, Roth [4] shows that

$$X=\sum_{s=1}^{q}{}^{\bullet}\,X_s,$$

where X_s is the same size as J_s, $s=1, \ldots, q$. If, further, each J_s is decomposed into a direct sum of companion matrices of the elementary divisors corresponding to λ_s one may also effect a conformal partitioning of the corresponding X_s. It is also clear that one may take $\lambda_s=0$ in examining the structure of X_s since $[J_s, X_s]_k$ remains invariant upon translation of J_s. We are thus reduced to considering the following situation in determining the dimension of $\mathfrak{A}_k(A)$: Let

$$U=\sum_{i=1}^{l}{}^{\bullet}\,U_{r_i}\qquad r_1\geq \ldots \geq r_l,$$

where U_{r_i} is an r_i-square auxiliary unit matrix, an unbroken line of 1's along the first super-diagonal 0's elsewhere, and suppose $[U, Y]_k=0$. Partition Y conformally with U, $Y=(Y_{ij})$ $i, j=1, \ldots, l$ where Y_{ij} is $r_i\times r_j$, and by (1.1)

$$\sum_{s=0}^{k}(-1)^s\binom{k}{s}U_{r_i}^{k-s}Y_{ij}U_{r_j}^{s}=0\qquad i, j=1, \ldots, l \tag{2.2}$$

is equivalent to $[U, Y]_k=0$. The problem then is to determine the number of arbitrary parameters in each Y_{ij}.

Equation (2.2) for a fixed i, j represents a linear transformation mapping Y_{ij} into 0, and with respect to a suitably chosen basis this transformation has the matrix representation T_{ij}^k where

$$T_{ij}=(I_{r_i}\otimes U_{r_j}-U'_{r_i}\otimes I_{r_j}). \tag{2.3}$$

To simplify the notation, put $r_i=n$, $r_j=m$ where it can be assumed without loss of generality that $n\geq m$. The similarity invariants of T_{ij} as computed by

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[4] W. E. Roth, On k-commutative matrices, Trans. Am. Math. Soc. **39**, 483 (1936).

Roth[5] are $f_1(x)=\ \ldots\ =f_{mn-m}(x)=1$, $f_{mn-m+p}(x)=x^{\Delta+2p-1}$, $p=1,\ \ldots,\ m$ and $\Delta=n-m$. Hence T_{ij} is similar to the direct sum of the companion matrices of these nontrivial similarity invariants,

$$T_{ij}\cong\sum_{p=1}^{m}{}^{\bullet}\ C(x^{\Delta+2p-1}). \tag{2.4}$$

The sizes of these companion matrices arranged in decreasing order are $\Delta+2m-1$, $\Delta+2m-3$, $\ldots$, $\Delta+3$, $\Delta+1$.

Now, if $k\geq\Delta+2p-1$, then $(C(x^{\Delta+2p-1}))^k=0$. If $k<\Delta+2p-1$, then

$$\rho(\{C(x^{\Delta+2p-1})\}^k)=\Delta+2p-1-k,$$

where ρ denotes rank.

Let η denote nullity.

LEMMA 1: (a) $\eta(T_{ij}^k)=km$ *if* $1\leq k<\Delta$,

(b) $\eta(T_{ij}^k)=km-\left(\frac{k-\Delta}{2}\right)^2+C$ *if* $\Delta\leq k<m+n-1$,

(c) $\eta(T_{ij}^k)=mn$ *if* $k\geq m+n-1$,

where C is 0 or 1/4 according as $(k-\Delta)$ is even or odd.

PROOF:

(a) $1\leq k<\Delta$.

Then

$$\rho(T_{ij}^k)=\sum_{p=1}^{m}(\Delta+2p-1-k)=mn-mk,$$

and

$$\eta(T_{ij}^k)=mk.$$

(b) $\Delta\leq k<n+m-1$.

Assume $k-\Delta$ is odd and observe that the size of the $(m-(k-\Delta+1)/2)$th companion matrix in (2.4) is

$$\Delta+2m-\left(2\left(m-\frac{(k-\Delta+1)}{2}\right)-1\right)=k+2,$$

and the size of the next companion matrix is k. Hence,

$$\rho(T_{ij}^k)=(\Delta+2m-1-k)+(\Delta+2m-3-k)+\ \ldots\ +(k+2-k).$$

The last term in this sum is the rank of the kth power of the $(m-(k-\Delta+1)/2)$th companion matrix in (2.4). Thus,

$$\rho(T_{ij}^k)=\left(\frac{2m-k+\Delta-1}{2}\right)\left(\frac{2m-k+\Delta+1}{2}\right)$$

$$=\left(m-\frac{(k-\Delta)}{2}\right)^2-\frac{1}{4}.$$

In case $k-\Delta$ is even, it is observed that the size of the $(m-(k-\Delta)/2)$th companion matrix in (2.4) is $k+1$. Also the size of the next companion matrix is $k-1$. Hence,

$$\rho(T_{ij}^k)=(\Delta+2m-1-k)+(\Delta+2m-3-k)+\ \ldots\ +(k+1-k)$$

$$=\left(m-\frac{(k-\Delta)}{2}\right)^2.$$

Hence, in either case

$$\eta(T_{ij}^k)=mn-\rho(T_{ij}^k)$$

$$=mn-\left(m-\frac{(k-\Delta)}{2}\right)^2+C$$

$$=mk-\left(\frac{k-\Delta}{2}\right)^2+C$$

where C is 0 or 1/4 depending on whether $k-\Delta$ is even or odd.

(c) $k\geq m+n-1$.

Then

$$\{C(x^{\Delta+2m-1})\}^k=0$$

and

$$\eta(T_{ij}^k)=mn.$$

THEOREM 1. *Assume A is N-square with distinct eigenvalues $\lambda_1,\ \ldots,\ \lambda_q$ and let $(x-\lambda_j)^{\epsilon_j}$ be the elementary divisors of A, $j=1,\ \ldots,\ q$, $\epsilon_1\geq\epsilon_2\geq\ \ldots\ \geq\epsilon_q$. Partition the integers $1,\ \ldots,\ q$ so that*

$$\epsilon_1=\ \ldots\ =\epsilon_{q_1}>\epsilon_{q_1+1}=\ \ldots$$

$$=\epsilon_{q_2}>\ \ldots\ >\epsilon_{q_{l-1}+1}=\ \ldots\ =\epsilon_{q_l}=\epsilon_q.$$

Then

(*i*) $\dim\mathfrak{A}_k(A)=kN-q\left(\frac{k^2}{4}-C\right)$ *if* $k<2\epsilon_q-1$,

(*ii*) $\dim\mathfrak{A}_k(A)=k\sum_{j=1}^{q_{l-1}}\epsilon_j+\sum_{j=q_{l-1}+1}^{q}\epsilon_j^2-q_{l-1}\left(\frac{k^2}{4}-C\right)$

if $2\epsilon_{q_l}-1\leq k<2\epsilon_{q_{l-1}}-1$,

(*iii*) $\dim\mathfrak{A}_k(A)=\sum_{j=1}^{q}\epsilon_j^2$ *if* $k\geq 2\epsilon_1-1$, *where C is 0 or 1/4 according as k is even or odd.*

[5] W. E. Roth, On direct product matrices, Bull. Am. Math. Soc. **40**, 461 (1934).

PROOF: Since there is only one elementary divisor of A for each eigenvalue of A it may be assumed, as in (2.1), that

$$A=\sum_{s=1}^{q}{}^{\bullet}\, J_s,$$

where $J_s=\lambda_s I_{\epsilon_s}+U_{\epsilon_s}$, U_{ϵ_s} the ϵ_s-square auxiliary unit matrix, $s=1, \ldots, q$. Then, if X is contained in $\mathfrak{A}_k(A)$,

$$X=\sum_{s=1}^{q}{}^{\bullet}\, X_s,$$

where X_s is ϵ_s-square, $s=1, \ldots, q$. By lemma 1 we know that the number of arbitrary parameters in X_s is (by putting $m=n=\epsilon_s$)

(b) $$n_s=\epsilon_s k-\frac{k^2}{4}+C \quad \text{if} \quad k<2\epsilon_s-1,$$

(c) $$\epsilon_s^2 \quad \text{if} \quad k\geq 2\epsilon_s-1.$$

Consider (*i*) first: $k<2\epsilon_q-1$. Then $k<2\epsilon_{q_j}-1$ $j=1, \ldots, l$, and hence X_s has n_s arbitrary parameters in it. There are $q_\sigma-q_{\sigma-1}$ values of s such that $n_s=n_{q_\sigma}$ ($q_0=0$ for convenience). Hence,

$$\begin{aligned}\dim \mathfrak{A}_k(A)&=\sum_{s=1}^{q} n_s=\sum_{\sigma=1}^{l}(q_\sigma-q_{\sigma-1})n_{q_\sigma}\\&=\sum_{\sigma=1}^{l}(q_\sigma-q_{\sigma-1})\left(k\epsilon_{q_\sigma}-\frac{k^2}{4}+C\right)\\&=k\sum_{\sigma=1}^{l}(q_\sigma-q_{\sigma-1})\epsilon_{q_\sigma}-q\left(\frac{k^2}{4}-C\right)\\&=k\sum_{j=1}^{q}\epsilon_j-q\left(\frac{k^2}{4}-C\right)\\&=kN-q\left(\frac{k^2}{4}-C\right).\end{aligned}$$

Next assume that (ii) $2\epsilon_{q_t}-1\leq k<2\epsilon_{q_{t-1}}-1$.

In this case $n_{q_\sigma}=\epsilon_{q_\sigma}^2$, $\sigma=t, \ldots, l$ and $n_{q_\sigma}=k\epsilon_{q_\sigma}-\frac{k^2}{4}+C$, $\sigma=1, \ldots, t-1$. Hence,

$$\begin{aligned}\dim \mathfrak{A}_k(A)&=\sum_{\sigma=1}^{t-1}(q_\sigma-q_{\sigma-1})n_{q_\sigma}+\sum_{\sigma=t}^{l}(q_\sigma-q_{\sigma-1})\epsilon_{q_\sigma}^2\\&=k\sum_{s=1}^{q_{t-1}}\epsilon_s+\sum_{s=q_{t-1}+1}^{q}\epsilon_s^2-q_{t-1}\left(\frac{k^2}{4}-C\right).\end{aligned}$$

If (*iii*) $k\geq 2\epsilon_1-1$, then $k\geq 2\epsilon_j-1$ for $j=1, \ldots, q$ and

$$\dim \mathfrak{A}_k(A)=\sum_{s=1}^{q}\epsilon_s^2.$$

In case there is more than one elementary divisor corresponding to a particular eigenvalue there does not seem to be any simple formula for $\dim \mathfrak{A}_k(A)$ in terms of the degrees of the elementary divisors of A. However, by repeated use of lemma 1, it is possible to compute $\dim \mathfrak{A}_k(A)$ for any particular A. For example, if

$$A=\begin{pmatrix}2&1&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2\end{pmatrix}\dot{+}\begin{pmatrix}2&1\\0&2\end{pmatrix}\dot{+}(2),$$

then $\dim \mathfrak{A}_k(A)=37$ for $k=3$.

Next are determined the largest and smallest values that $\dim \mathfrak{A}_k(A)$ may take on as A varies over E_q, the set of matrices with precisely q distinct eigenvalues, under the condition that $k\geq 2(N-q)+1$.

LEMMA 2. *If $\epsilon_1\geq \ldots \geq\epsilon_q$ are positive integers satisfying*

$$\sum_{j=1}^{q}\epsilon_j=N=qQ+R, \qquad 0\leq R<q,$$

then

$$R(Q+1)^2+(q-R)Q^2\leq \sum_{j=1}^{q}\epsilon_j^2\leq (N-q+1)^2+(q-1).$$

The lower bound is achieved for

$$\epsilon_1=\ldots=\epsilon_R=Q+1, \qquad \epsilon_{R+1}=\ldots=\epsilon_q=Q,$$

and the upper bound is achieved for

$$\epsilon_1=N-q+1 \quad \textit{and} \quad \epsilon_2=\ldots=\epsilon_q=1.$$

PROOF. The lower inequality is proved by induction on R. In case $R=0$, then $N/q=Q$, and if $\epsilon_1, \ldots, \epsilon_q$ are regarded as continuous variables, then $\sum_{j=1}^{q}\epsilon_j^2$ has a minimum for $\epsilon_j=Q$, $j=1, \ldots, q$. Now suppose the result is true for all remainders obtained by dividing N by q that are less than R. We first claim that there exists an integer $i<q$ such that $\epsilon_i>\epsilon_{i+1}$ and $\epsilon_i\geq Q+1$. Clearly the set of integers j such that $\epsilon_j\geq Q+1$ is nonempty otherwise (since $R>0$),

$$\sum_{j=1}^{q}\epsilon_j\leq qQ<N.$$

Let i be the largest integer j such that $\epsilon_j\geq Q+1$; then if i were q, $\epsilon_1\geq \ldots \geq\epsilon_q\geq Q+1$ and

$$\sum_{j=1}^{q}\epsilon_j\geq q(Q+1)>N.$$

Hence $i<q$ and from the definition of i, $\epsilon_i>\epsilon_{i+1}$. Let $\mu_j=\epsilon_j$, $j\neq i$ and $\mu_i=\epsilon_i-1$. Then $\mu_1\geq\ \ldots\ \geq\mu_q$, $\mu_1+\ \ldots\ +\mu_q=N-1=Qq+R-1$, and by induction

$$\sum_{j=1}^{q}\mu_j^2\geq(R-1)(Q+1)^2+(q-(R-1))Q^2=R(Q+1)^2 +(q-R)Q^2+(Q^2-(Q+1)^2).$$

Now

$$\sum_{j=1}^{q}\mu_j^2=\sum_{j=1}^{q}\epsilon_j^2-2\epsilon_i+1,$$

and thus,

$$\sum_{j=1}^{q}\epsilon_j^2\geq R(Q+1)^2+(q-R)Q^2+2(\epsilon_i-Q-1).$$

Since $\epsilon_i\geq Q+1$, the proof is complete. The upper bound is easily obtained.

THEOREM 2. *If $k\geq 2(N-q)+1$, then*

$$\min_{A\epsilon E_q\cap M_N}\dim\,\mathfrak{A}_k(A)=R(Q+1)^2+(q-R)Q^2$$

and

$$\max_{A\epsilon E_q\cap M_N}\dim\,\mathfrak{A}_k(A)=(N-q+1)^2+q-1,$$

where

$$N=qQ+R,\qquad 0\leq R<q.$$

PROOF. Let $A\epsilon E_q$ and suppose $\epsilon_1\geq\ \ldots\ \geq\epsilon_q$ are such integers that ϵ_j is the sum of the degrees of all elementary divisors of A corresponding to λ_j, $j=1,\ \ldots\ ,q$.

Then

$$\sum_{j=1}^{q}\epsilon_j=N,$$

and hence, $\epsilon_1\leq N-q+1$ and $2\epsilon_1-1\leq k$. Thus k is at least $2\mu-1$ where μ is the degree of any elementary divisor of A. From lemma 1 one may check in this case that $\min_{A\epsilon E_q\cap M_N}\dim\,\mathfrak{A}_k(A)$ may be evaluated by confining A to those matrices having precisely one elementary divisor for each eigenvalue. Hence $(x-\lambda_j)^{\epsilon_j}$ may be taken as the elementary divisors of A, $j=1,\ \ldots\ ,q$. By theorem 1 if $A\epsilon E_q$,

$$\dim\,\mathfrak{A}_k(A)=\sum_{j=1}^{q}\epsilon_j^2,$$

and the results follow from lemma 2.

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Selected Bibliography of Statistical Literature, 1930 to 1957: I. Correlation and Regression Theory

Lola S. Deming

(October 16, 1959)

This is the first in a series of bibliographies dealing with various specific subjects in the field of statistics. References and titles of important contributions in correlation and regression theory have been taken from technical journals published throughout the world since 1930.

Statistics per se is a relatively new science. Its literature has grown tremendously during the last several decades. Not the least of the statistician's difficulties is keeping abreast of this ever increasing abundance of writing even in his own particular area of the subject. Acquisition of more than a superficial acquaintance with statistical developments is rendered enormously more difficult by the fact that publications in this highly applied science appear in a myriad of journals, many of which are not readily accessible. This bibliography is intended to be of service by listing the most important contributions in statistical theory and methodology as judged by two prominent reviewing journals. The NBS Statistical Engineering Laboratory maintains a card file of abstracts taken from *Zentralblatt für Mathematik* for the years 1930 to 1939 and from *Mathematical Reviews* from 1940 onward. Each abstract has been assigned at least one subject category following the classification scheme of *Mathematical Reviews*. The present bibliography lists publications relating to correlation and regression theory. It is planned to follow this publication with similar ones dealing with other subject categories of statistics. As later abstracts appear and are classified and processed, it is hoped that they too will be published as supplements to the present bibliography. A considerable time lag is unavoidable, however.

The following information is extracted directly from the abstracts:

Author: The author's surname is followed by initials only. In the case of complicated surnames, the first capitalized word given in the reviewing journal is used. When the paper bears the names of more than one author, the journal reference appears with each author's name but the title of the paper is given with the senior author only. Multiple authorship is denoted by the symbol ♦ preceding the surname.

Title: Given exactly as in the reviewing journal. Titles of separately bound publications (books, reports, etc.) are italicized, and are followed by the publisher.

Reference to literature: The references were transcribed to punched cards thereby necessitating severe and unconventional abbreviations. The name of the journal in italics and the number of the volume in bold face, are followed by initial page number.

Date of publication: The next figure, in parentheses, shows the date when the article or book itself appeared.

M (for *Mathematical Reviews*) and *Z* (for *Zentralblatt für Mathematik*) are followed by the volume number and page number of the reviewing journal in which the abstract appears.

It is a pleasure to record the help of several members of the Statistical Engineering Laboratory. Dr. Churchill Eisenhart, who originally conceived the plans for this project, has always been ready with assistance and encouragement. Dr. Eugene Lukacs, now at the Catholic University of America, studied the alternatives for the subject classification and recommended the one used here. He also made innumerable selections from reviews not classified under the regular headings of probability and statistics. Dr. Norman Severo, now at the University of Buffalo, helped with the plans for mechanizing and transcribing the references in a form suitable for publication.

Adyanthaya, N. K., A formula for the spurious correlation between two functions of mutually uncorrelated variables, *J. Indian Math. Soc.* **19,** 161 (1932). Z **5**, 73

Aitken, A. C., On the statistical independence of quadratic forms in normal variates, *Biometrika* **37,** 93 (1950). M **12**, 35

♦Aitchison, J., (See S. J. Prais), *Rev. Inst. Internat. Statist.* **22,** 1 (1954).

Albert, A. A., The matrices of factor analysis, *Proc. Nat. Acad. Sci., U.S.A.* **30,** 90 (1944). M **5**, 209

Albert, A. A., The minimum rank of a correlation matrix, *Proc. Nat. Acad. Sci., U.S.A.* **30,** 144 (1944). M **6**, 6

Alter, D., Multiple correlation for prediction purposes, *J. Amer. Statist. Assoc.* **26,** 258 (1931). Z **3**, 18

Amato, V., Sulle correlazioni parziali, *Statistica. Milano* **10,** 341 (1950). M **12**, 430

♦Anderson, R. L., Distribution of the circular serial correlation coefficient for residuals from a fitted Fourier series, *Ann. Math. Statist.* **21,** 59 (1950). M **11**, 530

Anderson, R. L., Distribution of the serial correlation coefficient, *Ann. Math. Statist.* **13,** 1 (1942). M **4**, 22

Anderson, R. L., The problem of autocorrelation in regression analysis, *J. Amer. Statist. Assoc.* **49,** 113 (1954). M **15**, 728

♦Anderson, T. W., (See R. L. Anderson), *Ann. Math. Statist.* **21,** 59 (1950).

Anderson, T. W., The asymptotic distributions of the roots of certain determinantal equations, *J. Roy. Statist. Soc. B* **10,** 132 (1948). M **10**, 553

Andersson, W., Researches into the theory of regression, *Meddel. Lunds Astron. Observ.*, **64**, 1 (1932). Z **6**, 173

Andersson, W., On a new method of computing non-linear regression curves, *Ann. Math. Statist.* **5,** 81 (1934). Z **9**, 364

♦Andrews, F. C. (See Z. W. Birnbaum), *Psychometrika* **15,** 191 (1950).

Angoff, W. H., A note on the estimation of non-spurious correlations, *Psychometrika* **21,** 295 (1956). M **18**, 343

Arbey, L., Les erreurs d'observations considérées comme liées, *Bull. Astr.* **(2) 14,** 75 (1949). M **12**, 347

Arley, N., On the distribution of relative errors from a normal population of errors. A discussion of some problems in the theory of errors, *Danske Vid. Selsk. Math.-Fys. Medd.* **18,** No. 3 (1940). M **2**, 232

♦Babitz, M., A method for approximating the average intercorrelation coefficient by correlating the parts with the sum of the parts, *Psychometrika* **5,** 283 (1940). M **2**, 110

Bacon, H. M., A matrix arising in correlation theory, *Ann. Math. Statist.* **19,** 422 (1948). M **10**, 94

Bahadur, R. R., Sufficiency and statistical decision functions, *Ann. Math. Statist.* **25,** 423 (1954). M **16**, 154

Baker, G. A., Linear regression when the standard deviations of arrays are not all equal, *J. Amer. Statist. Assoc.* **36,** 500 (1941). M **3**, 173

Baker, G. A., Correlations between functions of variables, *J. Amer. Statist. Assoc.* **37,** 537 (1942). M **4**, 221

Banerjee, D. P., Note on the limit of correlation and regression coefficients in mingled records, *Math. Student* **9,** 155 (1941). M **4**, 104

Bartlett, M. S., Some aspects of the time-correlation problem in regard to tests of significance, *J. Roy. Statist. Soc.*, **98,** 536 (1935). Z **12**, 114

Bartlett, M. S., Further aspects of the theory of multiple regression, *Proc. Cambridge Philos. Soc.* **34,** 33 (1938). Z **18**, 158

Bartlett, M. S., The standard errors of discriminant function coefficients, *Suppl. J. Roy. Statist. Soc.* **6,** 169 (1939). M **1**, 248

Bartlett, M. S., The statistical significance of canonical correlations, *Biometrika* **32,** 29 (1941). M **2**, 235

♦Bates, G. E., Contributions to the theory of accident proneness. I. An optimistic model of the correlation between light and severe accidents, *Univ. California Publ. Statist.* **1,** 215 (1952). M **14**, 389

♦Bates, G. E., Contributions to the theory of accident proneness, II. True or false contagion, *Univ. California Publ. Statist.* **1,** 255 (1952). M **14**, 390

Bejar, J., Median regression and linear programming, *Trabajos Estadist.* **7,** 141 (1956). M **18**, 771

Bernstein, F., Über die Beobachtungsokonomie in der Statistik, *Assekuranz Jb.* **51,** 3 (1932) Z **4**, 256

Bernstein, F., Die mittleren Fehlerquadrate und Korrelationen der Potenzmomente und ihre Anwendung auf Funktionen der Potenzmomente, *Metron* **10,** Nr. 3,3 (1932). Z **5**, 256

Bernstein, F., Regression and correlation evaluated by a method of partial sums, *Ann. Math. Statist.* **8,** 77 (1937). Z **17**, 77

Bhargava, R. P., Test of significance for intra-class correlation when family sizes are not equal, *Sankhyā* **7,** 435 (1946). M **8**, 161

Bhattacharyya, A., The problem of regression in a statistical population admitting location parameters, *Bull. Inst. Internat. Statist.* **23,** 29, (1951). M **16**, 1040

Bhattacharyya, A., A note on the distribution of the sum of chi-squares, *Sankhyā* **7,** 27 (1945). M **7**, 131

♦Birnbaum, Z. W., On the effect of selection performed on some coordinates of a multi-dimensional population, *Psychometrika* **15,** 191 (1950). M **12**, 36

♦Bliss, C. I., (See W. G. Cochran), *Ann. Math. Statist.* **19,** 151 (1948).

Bojarski, A., Sur la corrélation géométrique, *Bull. Acad. Sci. URSS Sér. Math.* **5,** 159 (1941). M **3**, 173

Bonferroni, C., Un indice quadratico di concentrazione, *Atti. Secondo Cong. Un. Mat. Ital., Bologna* (1940). M **8**, 473

Bonnier, G., The four-fold table and the heterogeneity test, *Science* **96,** 13 (1942). M **4**, 26

Bose, P., On the reduction formulae for the incomplete probability integral of the multiple correlation coefficient of the second kind, *Science and Culture* **7,** 171 (1941). M **5**, 42

Brandner, F. A., Common elements, *Proc. Iowa Acad. Sci.* **63,** 528 (1956). M **18**, 425

♦Brennan, J. F., (See G. W. Housner), *Ann. Math. Statist.* **19,** 380 (1948).

Burgdörfer, F., Die Dynamik der deutschen Bevölkerungssent-wicklung, *Naturwissenschaften* **1932,** 367 (1932). Z **4**, 265

Burkhardt, F., Über Umkerhrung und Verknüpfund von statistischen Beziehungen, *Arch. Math. Wirtsch. u Sozialforschg* **7,** 1 (1941). Z **24**, 429

Cámara, S., Grundlagen der allgemeinen Theorie vielfacher Korrelation, *Rev. Mat. Hisp.-Amer.* **II 6,** 249 (1931). Z **5**, 367

Camp, K., Notes on interpolation. I. Subtabulation, or subdivision of intervals: A survey and appraisal of methods, *Trans. Actuar. Soc. Amer.* **38,** 16 (1937). Z **19**, 132

Campbell, J. T., The Poisson correlation function, *Proc. Edinburgh Math. Soc.* **II 4,** 18 (1934). Z **9**, 28

Carlson, P. G., A least squares interpretation of the bivariate line of organic correlation, *Skand. Aktuarietidskr* **39,** 7 (1956). M **18**, 771

Carlson, J. L., A study of the distribution of means estimated from small samples by the method of maximum likelihood for Pearson's type II curve, *Ann. Math. Statist.* **3,** 86 (1932). Z **4**, 265

Carter, A. H., The estimation and comparison of residual regressions where there are two or more related sets of observations, *Biometrika* **36,** 26 (1949). M **11**, 673

♦Castañs Carmargo, M., The logarithmic correlation, *An. Real Soc. Españ. Fis. Quim.* **52,** 117 (1956). M **18**, 79

Cavalli, L. L., Sulla correlazione media fra più caratteri in relazione alla biometria, *Metron* **15,** 173 (1949). M **11**, 445

Chandler, K. N., On a theorem concerning the secondary subscripts of deviation in multivariate correlation using Yule's notation, *Biometrika* **37,** 451 (1950). M **12**, 347

Charnley, F., Some properties of a composite, bivariate distribution in which the means of the component normal distributions are linearly related, *Canadian J. Research Sect. A.* **19,** 139 (1941). M **3**, 172

♦Cochran, W. G., Discriminant functions with covariance, *Ann. Math. Statist.* **19,** 151 (1948). M **10**, 50

Coleman, J. B., A coefficient of linear correlation based on the method of least squares and the line of best fit, *Ann. Math. Statist.* **3,** 79 (1932). Z **4**, 265

Cornish, E. A., The analysis of covariance in quasi-factorial designs, *Ann. Eugenics* **10,** 269 (1940). M **2**, 237

Cowan, D. R. G., A note on the coefficient of part correlation and of correlation of a dependent variable with all but one of a group of other variables, *J. Amer. Statist. Assoc.* **27,** 177 (1932). Z **4**, 358

Cowden, D. J., Correlation concepts and the Doolittle method, *J. Amer. Statist. Assoc.* **38,** 327 (1943). M **5**, 42

Craig, A. T., Variables correlated in sequence, *Bull. Amer. Math. Soc.* **39,** 129 (1933). Z **6**, 267

Craig, A. T., On the correlation between certain averages from small samples, *Ann. Math. Statist.* **4,** 127 (1933). Z **7**, 125

Craig, A. T., Bilinear forms in normally correlated variables, *Ann. Math. Statist.* **18,** 565 (1947). M **9**, 294

Craig, C. C., Sampling in the case of correlated observations, *Ann. Math. Statist.* **2,** 324 (1931). Z **5**, 256

Cramér, H., *Mathematical Methods of Statistics* (Princeton University Press, Princeton, N.J., 1946). M **8**, 39

♦Crawford, I., (*See* D. M. Hall) *Psychometrika* **10,** 93 (1945).

Cureton, E. E., Rank-biserial correlation, *Psychometrika* **21,** 287 (1956). M **18**, 343

Daly, J. F., A problem in estimation, *Ann. Math. Statist.* **12,** 459 (1941). M **3**, 174

Daniels, H. E., Grouping corrections for high autocorrelations, *Suppl. J. Roy. Statist. Soc.* **9,** 245 (1947). M **9**, 294

Daniels, H. E., The relation between measures of correlation in the universe of sample permutations, *Biometrika* **33,** 129 (1944). M **6**, 91

Daniels, H. E., A property of rank correlations, *Biometrika* **35,** 416 (1948). M **10**, 386

Daniels, H. E., Rank correlation and population models, *J. Roy. Statist. Soc. Ser. B* **12,** 171 (1950). M **12**, 725

Darmois, G., Sur la régression. Résultats nouveaux, Problèmes non résolus, *Colloq. Anal. Statist., Bruxelles*, pp. 9–23, G. Thone, Liège (1955). M **17**, 505

Darmois, G., Sur certaines formes de liaisons de probabilité, *Colloq. Internat. du Centre National de la Recherche Scientifique* **13**, pp. 19–21, Paris (1949). M **11**, 254

David, F. N., Correlations between χ^2 cells, *Biometrika* **35,** 418 (1948). M **10**, 465

♦Davidoff, M. D., A table for the rapid determination of the tetrachoric correlation coefficient, *Psychometrika* **18,** 115 (1953). M **14**, 995

de la Garza, A., Spacing of information in polynomial regression, *Ann. Math. Statist.* **25,** 123 (1954). M **15**, 725

Del Vecchio, E., Sulla dipendenza statistica, *Giorn. 1st Ital. Attuari* **4,** 235 (1933). Z **6**, 411

Delaporte, P., Une méthode d'analyse des corrélations et son application, *C. R. Acad. Sci. Paris* **209,** 142 (1939). M **1**, 63

Delaporte, P., Une méthode d'analyse des corrélations, *C. R. Acad. Sci., Paris* **208,** 1960 (1939). Z **21**, 340

Delaporte, P., Sur l'estimation des corrélations des caractères avec le facteur général et les facteurs de groupe et sur l'écart-type de cette estimation, en analyse factorielle, *C. R. Acad. Sci., Paris* **222,** 525 (1946). M **7**, 463

Demaria, G., Correlazioni economiche nel tempo, *Rend. Semin. Mat. Fis. Milano* **10,** 255 (1936). Z **17**, 367

Derksen, J. B. D., On some infinite series introduced by Tschuprow, *Ann. Math. Statist.* **10,** 380 (1939). M **1**, 152

Derksen, Y. B. D., Note on a remark of D. J. Struik on correlation coefficients, *Bull. Amer. Math. Soc.* **41,** 394 (1935). Z **12**, 29

Dieulefait, C. E., Grundlagen einer allgemeinen Korrelationstheorie, *An. Soc. Ci. Argent.* **115,** 20 (1933). Z **9**, 315

Dieulefait, C. E., Contribution à l'étude de la théorie de la corrélation, *Biometrika* **26,** 379 (1934). Z **11**, 33

Dieulefait, C. E., Verallgemeinerung des Abhängigkeitskoeffizienten einer Korrelation, *An. Soc. Ci. Argent.* **117,** 260 (1934). Z **11**, 33

Dieulefait, C. E., Sur la corrélation au sens des modes, *C. R. Acad. Sci., Paris* **200,** 1511 (1935). Z **11**, 219

Dixon, W. J., Further contributions to the problem of serial correlation, *Ann. Math. Statist.* **15**, 119 (1944). M **6,** 6

Dodd, E. L., Regression coefficients as means of certain ratios, *Amer. Math. Monthly* **44,** 306 (1937). Z **16,** 314

Dodd, E. L., Interior and exterior means obtained by the method of moments, *Ann. Math. Statist.* **9,** 153 (1938). Z **20,** 41

Dodd, E., La statistique mathématique. Certain coefficients of regression or trend associated with largest likelihood, *Actualités Scient. et Industr., No. 740,* pp. 5–14 (Paris, Hermann and Cie, 1938). Z **22,** 243

Dunlap, J. W., Note on the computation of tetrachoric correlation, *Psychometrika* **5,** 137 (1940). M **2,** 110

♦Durbin, J., Inversions and rank correlation coefficients, *J. Roy. Statist. Soc. Ser. B* **13,** 303 (1951). M **13,** 963

♦Durbin, J., Testing for serial correlation in least squares regression, *Biometrika* **37,** 409 (1950). M **12,** 512

Dwyer, P. S., The simultaneous computation of groups of regression equations and associated multiple correlation coefficients, *Ann. Math. Statist.* **8,** 224 (1937). Z **18,** 158

Dwyer, P. S., The computation of moments with the use of cumulative totals, *Ann. Math. Statist.* **9,** 288 (1938). Z **20,** 243

Dwyer, P. S., The skewness of the residuals in linear regression theory, *Ann. Math. Statist.* **12,** 104 (1941). M **2,** 233

Dwyer, P. S., The evaluation of multiple and partial correlation coefficients from the factorial matrix, *Psychometrika* **5,** 211 (1940). M **2,** 234

Dwyer, P. S., The calculation of correlation coefficients from ungrouped data with modern calculating machines, *J. Amer. Statist. Assoc.* **35,** 671 (1940). Z **24,** 430

Dwyer, P. S., Recent developments in correlation technique, *J. Amer. Statist. Assoc.* **37,** 441 (1942). M **4,** 164

Dwyer, P. S., A matrix presentation of least squares and correlation theory with matrix justification of improved methods of solution, *Ann. Math. Statist.* **15,** 82 (1944). M **5,** 245

Dwyer, P. S., Pearsonian correlation coefficients associated with least-squares theory, *Ann. Math. Statist.* **20,** 404 (1949). M **11,** 191

♦Dwyer, P. S., Symbolic matrix derivatives, *Ann. Math. Statist.* **19,** 517 (1948). M **10,** 278

Edgett, G. L., Multiple regression with missing observations among the independent variables, *J. Amer. Statist. Assoc.* **51,** 122 (1956). M **17,** 981

Elderton, W. P., Correzioni dei momenti quando la curva è simmetrica, *Giorn. 1st Ital. Attuari.* **9,** 145 (1938). Z **19,** 176

Elfving, G., Optimum allocation in linear regression theory, *Ann. Math. Statist.* **23,** 255 (1952). M **13,** 963

Eyraud, H., Sur une représentation nouvelle des corrélations continues, *C. R. Acad. Sci., Paris* **199,** 1356 (1934). Z **10,** 313

Eyraud, H., Correlazione e causalità, *Giorn. 1st Ital. Attuari.* **6,** 57 (1935). Z **11,** 219

Eyraud, H., Les principes de la mesure des corrélations, *Ann. Univ. Lyon Sect. A,* **1,** 30 (1936). Z **15,** 168

Fay, E. A., Grouping of observations for fitting regression curves, *Navord Rep. 1057* (U.S.N. Ord. Test Stat., Inyokern, Calif., 1948). M **11,** 608

Ferguson, T., On the existence of linear regression in linear structural relations, *Univ. California Publ. Statist.* **2,** 143 (1955). M **17,** 634

Féron, R., Mérites comparés des divers indices de corrélation, *J. Soc. Statist. Paris* **88,** 328 (1947). M **10,** 722

Féron, R., Information et régression, *C. R. Acad. Sci., Paris* **232,** 1636 (1951). M **12,** 843

Féron, R., De la régression, *C. R. Acad. Sci., Paris* **234,** 2143 (1952). M **13,** 853

♦Fieller, E. C., Corrigenda to: Correlated random normal deviates, *Biometrika* **43,** 496 (1956). M **18,** 422

Fischer, C. H., On correlation surfaces of sums with a certain number of random elements in common, *Ann. Math. Statist.* **4,** 103 (1933). Z **7,** 72

Fischer, C. H., On multiple and partial correlation coefficients of a certain sequence of sums, *Ann. Math. Statist.* **4,** 278 (1933). Z **8,** 124

Fischer, C. H., A sequence of discrete variables exhibiting correlation due to common elements, *Ann. Math. Statist.* **13,** 97 (1942). M **4,** 23

Fisher, R. A., The sampling distribution of some statistics obtained from nonlinear equations, *Ann. Eugenics* **9,** 238 (1939). M **1,** 248

Fisher, R. A., The negative binomial distribution, *Ann. Eugenics* **11,** 182 (1941) M **4,** 26

Fix, E., Distributions which lead to linear regressions, *Proc. Symp. Math. Stat. and Prob.,* pp. 79–91, (Univ. Cal. Press, Berkeley, 1949). M **10,** 553

Fréchet, M., Sur le coefficient, dit de corrélation, *C.R. Acad. Sci., Paris* **197,** 1268 (1933). Z **8,** 124

Fréchet, M., A general method of constructing correlation indices, *Proc. Math. Phys. Soc. Egypt* **3,** 13 (1946). M **8,** 592

Fréchet, M., Anciens et nouveaux indices de corrélation. Leur application au calcul des retards économiques, *Econometrica* **15,** 1 (1947). M **8,** 393

Fréchet, M., Anciens et nouveaux indices de corrélation: errata, *Econometrica* **15,** 374 (1947). M **9,** 194

Fréchet, M., Le coefficient de connexion statistique de Gini-Salvemini, *Mathematica, Timisoara* **23,** 46 (1948). M **10,** 50

Fréchet, M., Additional note on a general method of constructing correlation indices, *Proc. Math. Phys. Soc. Egypt* **3,** 73 (1948). M **11,** 191

Fréchet, M., Sur un essai infondé de sauver le coefficient classique dit de corrélation, *Rev. Inst. Internat. Statistique* **18,** 157 (1950). M **12,** 840

Fréchet, M., Sur les tableaux de corrélation dont les marges sont données, *Ann. Univ. Lyon Sect. A* **14,** 53 (1951). M **14,** 189

♦Friede, G., Zur Maximalkorrelation, *Z. Angew. Math. Mech.* **28,** 158 (1948). M **10,** 50

Frisch, R., *Statistical confluence analysis by means of complete regression systems,* (Oslo Univ. Okonomiske Inst. 1934). Z **11,** 219

♦Frisch, R., Statistical correlation and the theory of cluster types, *J. Amer. Statist. Assoc.* **26,** 375 (1931). Z **3,** 358

Frisch, R., Note on the phase diagram of two variates, *Econometrica* **5,** 326 (1937). Z **17,** 175

Frolow, V., Utilisation du coefficient de corrélation dans l'analyse harmonique, *C.R. Acad. Sci., Paris* **213,** 56 (1941). M **5,** 51

García Tranque, T., On the linear correlation coefficient, *Gaceta Mat.* **2,** 184 (1950). M **12,** 512

Garwood, F., The probability integral of the correlation coefficient in samples from a normal bivariate population, *Biometrika* **25,** 71 (1933). Z **7,** 124

Geary, R. C., Determination of linear relations between systematic parts of variables with errors of observation the variances of which are unknown, *Econometrica* **17,** 30 (1949). M **10,** 465

Geary, R. C., Non-linear functional relationship between two variables when one variable is controlled, *J. Amer. Statist. Assoc.* **48,** 94 (1953). M **14,** 776

Geary, R. C., The mathematical expectation of the mean square contingency when the attributes are mutually independent, *J. Roy. Statist. Soc.* **103,** 90 (1940). M **1,** 248

Geary, R. C., Inherent relations between random variables, *Proc. Roy. Irish Acad. Sect. A* **47,** 63 (1942). M **4,** 21

Gebelein, H., Das statistische Problem der Korrelation als Variations- und Eigenwertproblem und sein Zusammenhang mit der Ausgleichsrechnung, *Z. Angew. Math. Mech.* **21,** 364 (1941). M **4,** 104

Gebelein, H., Bemerkung über ein von W. Höffding vorgeschlagenes, massstabsinvariantes Korrelationsmass, *Z. Angew. Math. Mech.* **22,** 171 (1942). M **4,** 279

Gebelein, H., Verfahren zur Beurteilung einer sehr geringen Korrelation zwischen zwei statistischen Merkmalsreihen, *Z. Angew. Math. Mech.* **22,** 286 (1942). M **6,** 6

Geiringer, H., Bemerkung zur Wahrscheinlichkeit nicht unabhängiger Ereignisse, *Rev. Math. Union Interbalkan* **2,** 1 (1939). Z **21,** 424

Geiringer, H., Korrelationsmessung auf Grund der Summenfunktion, *Z. Angew. Math. Mech.* **13,** 121 (1933). Z **7,** 24

Geiringer, H., Korrelationsmodelle, *Z. Angew. Math. Mech.* **14,** 19 (1934). Z **8,** 370

Gengerelli, J. A., A simplified method for approximating multiple regression coefficients, *Psychometrika* **13,** 135 (1948). M **10,** 134

George, A., On the accuracy of the different approximations to the L_1-distribution, *Sankhyā* **7,** 20 (1945). M **7,** 131

Ghurye, S. G., Note on asymptotic estimation of parameters of an autoregressive process, *Ganita* **6,** 1 (1956). M **18,** 771

Gini, C., Sulla determinazione dell'indice di cograduazione, *Metron* **13,** 41 (1939). Z **22,** 374

Gini, C., Sur la théorie de la dispersion et sur la vérification et l'utilisation des schémas théoriques, *Metron* **14,** 3 (1940). M **11,** 191

Gini, C., Degli indici sintetici de correlazione e delle loro relazioni con l'indice interno di correlazione (intra class correlation coefficient) e con gli indici di correlazione tra serie di gruppi, *Metron* **14,** 241 (1941). M **8,** 474

Girshick, M. A., On the sampling theory of roots of determinantal equations, *Ann. Math. Statist.* **10,** 203 (1939). M **1,** 22

Glock, W. S., A rapid method of correlation for continuous time series, *Amer. J. Sci.* **240,** 437 (1942). M **4,** 27

♦Goheen, H. W., (See M. D. Davidoff) *Psychometrika* **18,** 115 (1953).

Goodman, T. P., Technique for approximate measurement of correlation coefficients, *J. Appl. Phys.* **27,** 773 (1956). M **17,** 1219

Grazia-Resi, B., Nuove ricerche sugli indici di cograduazione fra serie con termini uguali, *Statistica, Milano* **8,** 409 (1948). M **11,** 445

Griffin, H. D., On the coefficient of part correlation, *J. Amer. Statist. Assoc.* **27,** 298 (1932). Z **5,** 174

Grüneberg, H. J., Die multiple Faktoranalyse, *Mitteilungsblatt Math. Statist.* **4,** 9 (1952). M **13,** 963

Guest, P. G., Grouping methods in the fitting of polynominals to unequally spaced observations, *Biometrika* **43,** 149 (1956). M **17,** 1219

♦Guilford, J. P., On determining the reliability and significance of a tetrachoric coefficient of correlation, *Psychometrika* **7,** 243 (1942). M **4,** 165

Guldberg, A., Sur la loi de Bernoulli à deux variables et la corrélation, *C. R. Acad. Sci., Paris* **196,** 1634. (1933). Z **7,** 72

Guldberg, A., Zur Theorie der theoretischen Statistik, *Norsk Mat. Forenings Skr.* **II,** 41 (1933). Z **7,** 253

Guldberg, A., Sur les lois probabilités et la corrélation, *Ann. Inst. H. Poincaré* **5,** 159 (1935). Z **11,** 408

Guldberg, A., Über die Anwendung der Differenzengleichungen in der Wahrscheinlichkeitsrechnung, *Skand. Mat. Kongr.*, pp. 13–19, Stockholm (1935). Z **11,** 408

♦Gulliksen, H., Regression tests for several samples, *Psychometrika* **15,** 91 (1950). M **12,** 193

Guttman, L., "Best possible" systematic estimates of communalities, *Psychometrika* **21,** 273 (1956). M **18,** 343

Guttman, L., An approach for quantifying paired comparisons and rank order, *Ann. Math. Statist.* **17,** 144 (1946). M **8,** 41

Guttman, L., Multiple rectilinear prediction and the resolution into components, *Psychometrika* **5,** 75 (1940). M **2,** 234

Guttman, L., General theory and methods for matric factoring, *Psychometrika* **9,** 1 (1944). M **6,** 6

Guttman, L., The test-retest reliability of qualitative data, *Psychometrika* **11,** 81 (1946). M **8,** 41

Guttman, L., Image theory for the structure of quantitative variates, *Psychometrika* **18,** 277(1953). M **15,** 543

Haavelmo, T., The statistical implications of a system of simultaneous equations, *Econometrica* **11,** 1 (1943). M **4,** 220

♦Haldane, J. B. S., (See K. A. Kermack) *Biometrika* **37,** 30 (1950).

Haldane, J. B. S., Moments of r and χ^2 for a fourfold table in the absence of association, *Biometrika* **33,** 231 (1945). M **7,** 463

Haldane, J. B. S., The use of χ^2 as a test of homogeneity in a $(n\times 2)$-fold table when expectations are small, *Biometrika* **33,** 234 (1945). M **7,** 463

Haldane, J. B. S., A note on non-normal correlation, *Biometrika* **36,** 467 (1949). M **11,** 733

Halford, J. O., I. Normal frequencies of a one-dimensional crystal. II. An approximation to the lattice frequency distribution in isotropic solids, *J. Chem. Phys.* **19,** 1375 (1951). M **13,** 714

♦Hall, D. M., Factor analysis calculations by tabulating machines, *Psychometrika* **10,** 93 (1945). M **7,** 87

Halperin, M., Normal regression theory in the presence of intra-class correlation, *Ann. Math. Statist.* **22,** 573 (1951). M **13,** 261

Hannan, E. J., Exact tests for serial correlation in vector processes, *Proc. Cambridge Philos. Soc.* **52,** 482 (1956). M **18,** 520

Hannan, E. J., The asymptotic powers of certain tests based on multiple correlations, *J. Roy. Statist. Soc. Ser. B* **18,** 227 (1956). M **18,** 682

Harkin, B., The expected error of a least-squares solution of location from direction-finding equipment, *Austral, J. Appl. Sci.* **7,** 263 (1956). M **18,** 606

Harley, B. I., A note on the probability integral of the correlation coefficient, *Biometrika* **41,** 278 (1954). M **15,** 971

Harley, B. I., Some properties of an angular transformation for the correlation coefficient, *Biometrika* **43,** 219 (1956). M **17,** 981

Harman, H. H., On the rectilinear prediction of oblique factors, *Psychometrika* **6,** 29 (1941). M **2,** 234

Hartley, H. O., The estimation of non-linear parameters by "internal least squares", *Biometrika* **35,** 32 (1948). M **10,** 50

Harzer, P., Tabellen für alle statistischen Zwecke in Wissenschaft und Praxis, insbesondere zur Ableitung des Korrelations-Koeffizienten, *Abh. Bayer. Mat.-Nat. Kl.* **21,** 1 (1933). Z **8,** 123

Hayes, S. P., Tables of the standard error of tetrachoric correlation coefficient, *Psychometrika* **8,** 193 (1943). M **5,** 42

Hayes, S. P., Diagrams for computing tetrachoric correlation coefficients from percentage differences, *Psychometrika* **11,** 163 (1946). M **8,** 41

Hekker, Th., Construction of chance ellipses in a correlation diagram, *Statistica, Leiden* **1,** 203 (1947). M **11,** 259

Herdan, G., The logical and analytical relationship between the theory of accidents and factor analysis, *J. Roy. Statist. Soc.* **106,** 125 (1943). M **5,** 209

Hirschfeld, H. O., A connection between correlation and contingency, *Proc. Cambridge Philos. Soc.* **31,** 520 (1935). Z **12,** 363

Hoel, P. G., A significance test for component analysis, *Ann. Math. Statist.* **8,** 149 (1937). Z **17,** 365

Hoel, P. G., The errors involved in evaluating correlation determinants, *Ann. Math. Statist.* **11,** 58 (1940). M **1,** 248

Hoel, P. G., On statistical coefficients of likeness, *Univ. Cal. Publ. Math.* **2,** 1 (1944). M **6,** 6

Hoel, P. G., Confidence regions for linear regression, *Proc. Symp. Math. Stat. and Prob.*, pp. 77–81, (Univ. Cal. Press, Berkeley, 1951). M **13,** 481

Höffding, W., Maszstabinvariante Korrelationstheorie, *Schr. Math. Inst. U. Inst. Angew. Math. Univ., Berlin,* **5,** 181 (1940). M **3,** 5

Höffding, W., Stochastische Abhängigkeit und funktionaler Zusammenhang, *Skand. Aktuarietidskr,* **25,** 200 (1942). M **7,** 212

Hoffmann, T. A., Some investigations in the field of the theory of solids. II. Linear chains of different atoms. Binary systems, *Acta Phys. Acad. Sci. Hungaricae* **1,** 175 (1951). M **13,** 714

Holzinger, K. J., Factoring test scores and implications for the method of averages, *Psychometrika* **9,** 155 (1944). M **6,** 92

Hope, A. E., A formula in the theory of correlation, *Math Notes* Nr 28, XIV (1933). Z **8,** 25

Horst, P., A short method for solving for a coefficient of multiple correlation, *Ann. Math. Statist.* **3,** 40 (1932). Z **4,** 265

Horst, P., A general method for evaluating multiple regression constants, *J. Amer. Statist. Assoc.* **27,** 270 (1932). Z **5,** 256

Horst, P., A non-graphical method for transforming an arbitrary factor matrix into a simple structure factor matrix, *Psychometrika* **6,** 79 (1941). M **2,** 234

♦Hotelling, H., Rank correlation and tests of significance involving no assumption of normality, *Ann. Math. Statist.* **7,** 29 (1936). Z **14,** 29

Hotelling, H., New light on the correlation coefficient and its transforms, *J. Roy. Statist. Soc. Ser. B* **15,** 193 (1953) M **15,** 728

♦Housner, G. W., The estimation of linear trends, *Ann. Math. Statist.* **19,** 380 (1948). M **10,** 201

Hsu, P. L., On the distribution of roots of certain determinantal equations, *Ann. Eugenics* **9,** 250 (1939). M **1,** 248

Hsu, P. L., On the limiting distribution of the canonical correlations, *Biometrika* **32,** 38 (1941). M **2,** 234

Hsu, P. L., Canonical reduction of the general regression problem, *Ann. Eugenics* **11,** 42 (1941). M **3,** 8

Hudimoto, H., Note on fitting a straight line when both variables are subject to error and some applications, *Ann. Inst. Statist. Math., Tokyo* **7,** 159 (1956). M **18,** 242

Huntington, E. V., An improved equal-frequency map of the normal correlation surface, using circles instead of ellipses, *Proc. Nat. Acad. Sci. U.S.A.* **18,** 465 (1932). Z **4,** 301

Huntington, E. V., An improved equal-frequency map of the normal correlation surface, using circles instead of ellipses, *J. Amer. Statist. Assoc.* **27,** 251 (1932). Z **5,** 174

Hussain, Q. M., A note on interaction, *Sankhyā* **6,** 321 (1943). M **5,** 126

Huttly, N. A., The fitting of regression curves with autocorrelated data, *Biometrika* **43,** 468 (1956). M **18,** 522

♦Ignatius, J. G. W., (See E. van der Laan), *Statistica, Leiden* **1,** 219 (1947).

James, G. S., The comparison of several groups of observations when the ratios of the population variances are unknown, *Biometrika* **38,** 324 (1951). M **13,** 762

Janne d'Othée, H., Sur les polynômes employés dans les regréssions de divers degrés en statistique, *Bull. Soc. Roy. Sci. Liège* **25,** 251 (1956). M **18,** 242

Jaspen, N., Serial correlation, *Psychometrika* **11,** 23 (1946). M **7,** 462

Jones, H. E., Some geometrical considerations in the general theory of fitting lines and planes, *Metron* **13,** 21 (1937). Z **16,** 172

Jones, H. E., The nature of regression functions in the correlation analysis of time series, *Econometrica* **5,** 305 (1937). Z **17,** 175

Jones, H. L., Linear regression functions with neglected variables, *J. Amer. Statist. Assoc.* **41,** 356 (1946). M **8,** 161

Jordan, C., La statistique mathématique. Critique de la corrélation au point de vue de la probabilité, *Actualités Scient. et Indust. No. 740,* pp. 15–33. Paris, Hermann and Cie (1938). Z **22,** 243

Jowett, G. H., Least squares regression analysis for trend-reduced time series, *J. Roy. Statist. Soc. Ser. B* **17,** 91 (1955). M **17,** 505

K. N. M., A note on correlation between two unbiased estimators, *Calcutta Statist. Bull.* **4,** 72 (1952). M **14,** 190

Kärsna, A., Vereinfachte Methoden zur Berechnung des Korrelationskoeffizienten bei normaler Korrelation, *Acta et Comment. Univ. Tartu* **A 29,** 1 (1936). Z **14,** 321

Kašanin, R., Le coefficient d'approximation moyenne et le coefficient de corrélation, *Acad. Serbe Sci. Publ. Inst. Math.* **1,** 71 (1947). M **10,** 552

Keeping, E. S., A significance test for exponential regression, *Ann. Math. Statist.* **22,** 180 (1951). M **12,** 842

Kelley, T. L., An unbiased correlation ratio measure, *Proc. Nat. Acad. Sci. U.S.A.* **21,** 554 (1935). Z **12,** 217

Kemp, W. B., Some methods for statistical analysis, *J. Amer. Statist. Assoc.* **29,** 147 (1934). Z **9,** 221

Kendall, M. G., Rank and product-moment correlation, *Biometrika* **36,** 177 (1949). M **11,** 673

Kendall, M. G., Proof of relations connected with the tetrachoric series and its generalizations, *Biometrika* **32,** 196 (1941). M **3,** 173

Kendall, M. G., The relationship between correlation formulae and elliptic functions, *J. Roy. Statist. Soc.* **104,** 281 (1941). M **3,** 173

Kendall, M. G., Partial rank correlation, *Biometrika* **32,** 277 (1942). M **4,** 22

Kendall, M. G., The treatment of ties in ranking problems, *Biometrika* **33,** 239 (1945). M **8,** 41

Kendall, M. G., The variance of τ when both rankings contain ties, *Biometrika* **34,** 297 (1947). M **9,** 453

Kendall, M. G., Regression, structure and functional relationship. II, *Biometrika* **39,** 96 (1952). M **14,** 66

♦Kermack, K. A., Organic correlation and allometry, *Biometrika* **37,** 30 (1950). M **12,** 430

♦Keys, N., *Psychometrika* **5,** 283 (1940). (See M. Babitz.)

Khintchine, A., Über stationäre Reihen zufälliger Variablen, *Rec. Math. Moscou* **40,** 124 (1933). Z **7,** 313

Khintchine, A., Korrelationstheorie der stationären stochastischen Prozesse, *Math. Ann.* **109,** 604 (1934). Z **8,** 368

Kimball, B. F., A multiple group least squares' problem and the significance of the associated orthogonal polynomials, *J. Amer. Statist. Assoc.* **48,** 320 (1953). M **14,** 1127

♦Kimball, A. W., *Biometrics* **6,** 259 (1950). (See D. F. Votaw.)

Kingston, J., *A Theoria da Inducão Estatistica*, (Instit. Brasil Geog. e. Estatíst, Rio de Janeiro, 1945). M **7,** 463

Kirkham, W. J., Note on the derivation of the multiple correlation coefficient, *Ann. Math. Statist.* **8,** 68 (1937). Z **16,** 171

Knudsen, L. F., Interdependence in a series, *J. Amer. Statist. Assoc.* **35,** 507 (1940). Z **24,** 430

Koeppler, H., Das Fehlergesetz des Korrelationskoeffizienten und andere Wahrscheinlichkeitsgesetze der Korrelationstheorie, *Metron* **12,** 105 (1935). Z **11,** 409

Koller, S., Die Analyse der Abhängigkeitsverhältnisse in zwei Korrelations-systemen, *Metron* **12,** 73 (1936). Z **15,** 33

Konijn, H. S., On certain classes of statistical decision procedures, *Ann. Math. Statist.* **24,** 440 (1953). M **16,** 499

Koopmans, T., Serial correlation and quadratic forms in normal variables, *Ann. Math. Statist.* **13,** 14 (1942). M **4,** 22

Koopmans, T. C., Statistical Inference in Dynamic Economic Models, *Cowles Commission Monograph No.* **10** (J. Wiley, N.Y., 1950). M **12,** 431

Kosambi, D. D., Correlation and time series, *Current Sci.* **10,** 372 (1941). M **3,** 175

Krawtchouk, M., Sur les écarts moyens des coefficients de corrélation et de régression, *J. Inst. Math. Acad. Sci. Ukraine* Nr. **3/4,** 121 (1935). Z **12,** 113

Kreis, H., Lineare Abhängigkeit und Äquivalenz von Punktsystemen, *Mitt. Verein, Schweiz. Versich.-Math.* **46,** 169 (1946). M **8,** 393

Krishnaswami Ayyangar, A. A., Correlation by the method of factors, *Proc. Indian Acad. Sci. Sect. A* **6,** 71 (1937). Z **17,** 273

♦Krishnaswami, G. V., On a representation of the coefficients of correlation, by the constants of a spherical triangle, *J. Annamalai Univ.* **3,** 189 (1934). Z **10,** 214

Krull, W., Zur Korrelationstheorie zweidimensionaler Merkmale, *Mitteilungsblatt Math. Statist.* **3,** 15 (1951). M **13,** 762

Krull, W., Korrelationstheorie mehrdimensionaler Merkmale, *Mitteilungsblatt Math. Statist.* **3,** 185 (1951). M **13,** 762

Kruskal, W. H., On the uniqueness of the line of organic correlation, *Biometrics* **9,** 47 (1953). M **14,** 890

Kullback, S., A note on the multiple correlation coefficient, *Metron* **12,** 67 (1936). Z **15,** 118

Kuzmin, R. O., Sur la loi de distribution du coefficient de correlation dans les tirages d'un ensemble normal, *C.R. Acad. Sci. URSS,* **22,** 298 (1939). Z **21,** 41

Lakshmanamurti, M., Coefficient of association between two attributes in statistics, *Proc. Indian Acad. Sci. Sect. A* **22,** 123 (1945). M **7,** 212

Lancaster, H. O., The derivation and partition of χ^2 in certain discrete distributions, *Biometrika* **36,** 117 (1949). M **12,** 191

Landenna, G., La dissomiglianza, *Statistica. Bologna* **16,** 21 (1956) M **18,** 242

Latycheva, K., Sur quelques schémas d'urnes pour l'explication d'une corrélation normale, *J. Cycle Math. Acad. Sci. Ukraine* **1,** 71 (1934). Z **10,** 174

Leslie, P. H., A simple method of calculating the exact probability in 2×2 contingency tables with small marginal totals, *Biometrika* **42,** 522 (1955). M **17,** 169

Lev, J., The point biserial coefficient of correlation, *Ann. Math. Statist.* **20,** 125 (1949). M **10,** 465

♦Lewis, T., (See E. C. Fieller) *Biometrika* **43,** 496 (1956).

Li, C. C., The concept of path coefficient and its impact on population genetics, *Biometrics* **12,** 190 (1956). M **18,** 630

Lindblad, T., Zur Theorie der Korrelation bei mehrdimensionalen zufälligen Variablen, *Acta Soc. Sci. Fennicae,* **2,** 1 (1937). Z **18,** 76

Lindblom, S. G., On the connection between tests of significance for correlation coefficients and for differences between means, *Skand. Aktuarietidskr* **29,** 12 (1946). M **8,** 42

Lindley, D. V., Regression lines and the linear functional relationship, *Suppl. J. Roy. Statist. Soc.* **9,** 218 (1947). M **9,** 363

Lindley, D. V., A regression problem, *Proc. Cambridge Philos. Soc.* **47,** 337 (1951). M **12,** 842

Lord, F. M., Alignment chart for calculating the fourfold point correlation coefficient, *Psychometrika* **9,** 41 (1944). M **5,** 208

Lovera, G., Un'applicazione del coefficiente di correlazione alle medie statistiche, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat.* **77,** 341 (1942). M **7,** 463

Lovera, G., Metodo abbreviato di calcolo delle caratteristiche di una correlazione multipla, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat.* **80,** 194 (1945). M **8,** 473

Lyerly, S. B., The average Spearman rank correlation coefficient, *Psychometrika* **17,** 421 (1952). M **14,** 665

♦Lyons, T. C., (See J. P. Guilford). *Psychometrika* **7,** 243 (1942).

♦Macphail, M. S. (See P. S. Dwyer). *Ann. Math. Statist.* **19,** 517 (1948).

Madow, W. G., Limiting distributions of quadratic and bilinear forms, *Ann. Math. Statist.* **11,** 125 (1940). M **1,** 341

Madow, W. G., Note on the distribution of the serial correlation coefficient, *Ann. Math. Statist.* **16,** 308 (1945). M **7,** 131

Martin, D., On the radial error in a Gaussian elliptical scatter, *Philos. Mag.* **37,** 636 (1946). M **8,** 523

Masuyama, M., Correlation between tensor quantities, *Proc. Phys. Math. Soc. Japan* **21,** 638 (1939). M **1,** 151

Masuyama, M., On the meaning of the symmetric correlation coefficient between vector sets, *Proc. Phys. Math. Soc. Japan* **22,** 579 (1940). M **2,** 110

Masuyama, M., On the subdependency, *Proc. Phys. Math. Soc. Japan* **22,** 855 (1940). M **2,** 233

Masuyama, M., The variance tensor of vector set and a nature of the symmetric correlation coefficient, *Proc. Phys. Math. Soc. Japan* **22,** 858 (1940). M **2,** 234

Masuyama, M., On the characteristic values of the correlation tensor and a new measure of correlation between vector quantities, *Proc. Phys. Math. Soc. Japan* **23,** 199 (1941). M **3,** 6

Masuyama, M., The normal law of frequency for vector quantities, *Proc. Phys. Math. Soc. Japan* **23,** 196 (1941). M **3,** 6

Masuyama, M., The totally orthonormalised vector set and the normal form of correlation tensor, *Proc. Phys. Math. Soc. Japan* **23,** 346 (1941). M **3,** 6

Masuyama, M., The standard error of the mean vector, *Proc. Phys. Math. Soc. Japan* **23,** 194 (1941). M **3,** 6

Masuyama, M., The mean angle between two vector sets, *Proc. Phys. Math. Soc. Japan* **23,** 351 (1941). M **3,** 6

Masuyama, M., Correlation coefficient between two sets of complex vectors, *Proc. Phys. Math. Soc. Japan* **23,** 918 (1941). M **7,** 316

Masuyama, M., On the significance test of the additive correlation coefficient, *Proc. Phys. Math. Soc. Japan* **23,** 1016 (1941). M **7,** 316

Matérn, B., Independence of non-negative quadratic forms in normally correlated variables, *Ann. Math. Statist.* **20,** 119 (1949). M **10,** 553

Matsumura, S., Bemerkung zu Korrelations Theorie, *J. Osaka Inst. Sci. Tech. Part I* **3,** 33 (1951). M **15,** 45

McIntyre, F., Automatic checks in correlation analysis, *J. Amer. Statist. Assoc.* **32,** 119 (1937). Z **16,** 67

♦Medina e. Isabel, M., (See M. Castañs Camargo), *An. Real. Soc. Espan. Fis. Quim.* **52,** 117 (1956).

Merzrath, E., Anpassung von Flächen an zweidimensionale Kollektivgegenstände und ihre Auswertung für die Korrelationstheorie, *Metron* **11,** 103 (1933). Z **7,** 356

Milicer Gruzewska, H., The coefficient of correlation a posteriori of equivalent variables, *Soc. Sci. Lett. Varsovie C. R. Cl. III. Sci. Math. Phys.* **39,** 3(1946). M **11,** 374

Mine, A., Estimation of linear regression coefficients in time series. (A note on the generalization of minimum distance method.) *Ann. Inst. Statist. Math. Tokyo* **6,** 181 (1955). M **17,** 505

Miner, J. R., The standard error of a multiple regression equation, *Ann. Math. Statist.* **2,** 320 (1931). Z **4,** 264

Mitropolsky, A. On the multiple non-linear correlation equations, *Bull. Acad. Sci. URSS Ser. Math.* **1939,** 399 (1939). M **1,** 345

Mitropolsky, A. Berechnung ein fachler Regressionsgleichungen, *J. Techn. Physics, Leningrad* **10,** 1227 (1940). Z **24,** 161

Mitropolskii, A. K., Ordinary correlation equations, *Uspehi Matem. Nauk.* **4,** 142 (1949). M **11,** 259

Montessus de Ballore, R., *La méthode de corrélation suivie de la table des carrés des nombres entiers de 1 à 1,000,* Gauthier-Villars et Cie, Paris (1932). Z **5,** 405

Moran, P. A. P., A test for the serial independence of residuals, *Biometrika* **37,** 178 (1950). M **12,** 38

Moran, P. A. P., Recent developments in ranking theory, *J. Roy. Statist. Soc. Ser. B* **12,** 153 (1950). M **12,** 725

Moran, P. A. P., On the method of paired comparisons, *Biometrika* **34,** 363 (1947). M **9,** 363

Moran, P. A. P., Rank correlation and permutation distributions, *Proc. Cambridge Philos. Soc.* **44,** 142 (1948). M **9,** 263

Moran, P. A. P., The distribution of the multiple correlation coefficient, *Proc. Cambridge Philos. Soc.* **46,** 521 (1950). M **11,** 732

Moritz, R. E., A new theory of depreciation of physical assets, *Ann. Math. Statist.* **3,** 108 (1932). Z **4,** 265

Morrison, J. T., Note on the maximum correlation coefficient between two series whose values have been determined at equal intervals, *Philos. Mag. VII* **24,** 240 (1937). Z **17,** 77

Morrison, J. T., Note on the correlation of time-series, *Philos. Mag. VII* **18,** 545 (1934). Z **10,** 72

Mosak, J. L., On the simultaneous determination of the elementary regressions and their standard errors in subsets of variables, *J. Amer. Statist. Assoc.* **32,** 517 (1937). Z **17,** 273

♦ Mourier, E., (See C. Rothchild) *C. R. Acad. Sci. Paris* **225,** 1117 (1947).

♦Mudgett, B. D., (See R. Frisch) *J. Amer. Statist. Assoc.*, **26,** 375 (1931).

Muhsam, H. V., A probability approach to ties in rank correlation, *Bull. Res. Council Israel* **3,** 321 (1954). M **16,** 731

Münzner, H., Grundbegriffe und Probleme der Korrelationsrechnung, *Deutsche Math.* **1,** 290 (1936). Z **14,** 122

♦Münzner, H., (See G. Friede) *Z. Angew. Math. Mech.* **28,** 158 (1948).

Nair, A. N. K., Distribution of Student's "t" and the correlation coefficient in samples from non-normal populations, *Sankhyā* **5,** 383 (1941). M **4,** 164

Nair, U. S., The application of the moment function in the study of distribution laws in statistics, *Biometrika* **30,** 274 (1939). Z **20,** 148

Nair, U. S., Probability statements regarding the ratio of standard deviations and correlation coefficient in a bivariate normal population, *Sankhyā* **5,** 151 (1941). M **4,** 164

Narain, R. D., A new approach to sampling distributions of the multivariate normal theory, *J. Indian Soc. Agric. Statist.* **1,** 137 (1948). M **11,** 607

Neyman, J., Existence of consistent estimates of the directional parameter in a linear structural relation between two variables, *Ann. Math. Statist.* **22,** 497 (1951). M **13,** 481

♦Neyman, J., (See G. E. Bates) *Univ. California Publ. Statist.* **1,** 215 (1952).

♦Neyman, J., (See G. E. Bates) *Univ. California Publ. Statist.* **1,** 255 (1952).

Obuchoff, A. M., Eine Korrelationstheorie der Vektoren, *Učhenye Zapiski Moskov. Gos. U. Matemat.* **45,** 73 (1940). M **3,** 6

Ogawara, M., A note on the test of serial correlation coefficients, *Ann. Math. Statist.* **22,** 115 (1951). M **12,** 726

Olds, E. G., A moment-generating function which is useful in solving certain matching problems, *Bull. Amer. Math. Soc.* **44,** 407 (1938). Z **19,** 129

♦Olmstead, P. S., A corner test for association, *Ann. Math. Statist.* **18,** 495 (1947). M **9,** 294

Orcutt, G. H., A new regression analyser, *J. Roy. Statist. Soc. Ser. A* **111,** 54 (1948). M **10,** 404

Ottestad, P., The exponential frequency function and frequency distributions, *Metron* **13,** 51 (1937). Z **16,** 171

Ottestad, P., On Bernoullian, Lexis, Poisson and Poisson-Lexis series, *Skand. Aktuarietidskr* **26,** 15 (1943). M **7,** 211

♦Pabst, M. R., (See H. Hotelling) *Ann. Math. Statist.*, **7,** 29 (1936).

Patterson, H. D., A simple method for fitting an asymptotic regression curve, *Biometrics* **12,** 323 (1956). M **18,** 344

♦Paulson, E., (See Z. W. Birnbaum) *Psychometrika* **15,** 191 (1950).

Pearson, K., Remarks on Professor Steffensen's measure of contingency, *Biometrika* **26,** 255 (1934). Z **9,** 221

Pearson, E. S., The test of significance for the correlation coefficient, *J. Amer. Statist. Assoc.*, **26,** 128 (1931). Z **2,** 281

Pearson, K., Some properties of "Student's" z: Correlation, regression and scedasticity of z with the mean and standard deviation of the sample, *Biometrika* **23,** 1 (1931). Z **3,** 358

♦Pearson, E. S., (See E. C. Fieller) *Biometrika* **43,** 496 (1956).

Pearson, K., Appendix to a paper by Dr. Wishart. Tables of the mean value and square standard deviation of the square of a multiple correlation coefficient, *Biometrika* **22,** 362 (1931). Z **2,** 41

Picard, H. C., Eine allgemeine Theorie der mehrdimensionalen Korrelation, *Mitteilungsblatt Math. Statist.* **3,** 103 (1951). M **13,** 665

Pillai, K. C. S., Confidence interval for the correlation coefficient, *Sankhyā* **7,** 415 (1946). M **8,** 283

Pitman, E. J. G., A note on normal correlation, *Biometrika* **31,** 9 (1939). Z **21,** 339

Pitman, E. J. G., Tests of hypotheses concerning location and scale parameters, *Biometrika* **31,** 200 (1939). M **1,** 63

Platt, J. R., A mechanical determination of correlation coefficients and standard deviations, *J. Amer. Statist. Assoc.* **38,** 311 (1943). M **5,** 42

Pompilj, G., Sulla regressione, *U. Roma Ist Naz. Alta Mat. Rend. Matie Appl.* **5,** 186 (1946). M **9,** 150

♦Prais, S. J., The grouping of observations in regression analysis, *Rev. Inst. Internat. Statist.* **22,** 1 (1954). M **17,** 277

Quenouille, M. H., Some results in the testing of serial correlation coefficients, *Biometrika* **35,** 261 (1948). M **10,** 387

Quenouille, M. H., The evaluation of probabilities in a normal multivariate distribution, with special reference to the correlation ratio, *Proc. Edinburgh Math. Soc.* **8,** 95 (1949). M **11,** 673

Quenouille, M. H., The joint distribution of serial correlation coefficients, *Ann. Math. Statist.* **20,** 561 (1949). M **12,** 118

Quenouille, M. H., *Associated measurements*, (Academic Press, N.Y., 1952). M **14,** 568

Quensel, C. E., A method of determining the regression curve when the marginal distribution is of the normal logarithmic type, *Ann. Math. Statist.* **7,** 196 (1936). Z **15,** 309

Quensel, C. E., The distribution of the second moment and of the correlation coefficient in samples from populations of type A, *Lunds Univ. Årsskr, N. F.* **34,** 1 (1938). Z **18,** 320

♦Rafferty, J. A., (See D. F. Votaw) *Biometrics* **6,** 259 (1950).

Raghavan Nair, K., A note on the existence of lagging correlations between two random series, *J. Roy. Statist, Soc.* **99,** 559 (1936). Z **14,** 358

Rao, C. R., On bivariate correlation surfaces, *Science and Culture* **8,** 236 (1942). M **5,** 126

Rao, C. R., Note on a problem of Ragnar Frisch, *Econometrica* **15,** 245 (1947). M **8,** 592

Rao, C. R., A correction to "Note on a problem of Ragnar Frisch", *Econometrica* **17,** 212 (1949). M **11,** 259

Rao, C. R., A note on the distribution of $D^2_{p+q}-D^2_p$ and some computational aspects of D^2 statistic and discriminant function, *Sankhyā* **10,** 257 (1950). M **12,** 428

Rao, C. R., Discriminant functions for genetic differentiation and selection, *Sankhyā* **12,** 229 (1953). M **15,** 543

Rasch, G., On simultaneous factor analysis in several populations, *Uppsala Symp. Psychol. Factor Anal.* pp. 65–71; (Munksgaard, Copenhagen 1953). M **16,** 385

Reiersol, O., A note on the signs of gross correlation coefficients and partial correlation coefficients, *Biometrika* **43,** 480 (1956). M **18,** 344

Reiersol, O., Identifiability of a linear relation between variables which are subject to error, *Econometrica* **18,** 375 (1950). M **12,** 347

Reiersol, O., On the identifiability of parameters in Thurstone's multiple factor analysis, *Psychometrika* **15,** 121 (1950). M **12,** 38

Richards, H. I., Analysis of the spurious effect of high intercorrelation of independent variables on regression and correlation coefficients, *J. Amer. Statist. Assoc.* **26,** 21 (1931). Z **2,** 41

Richter, H., Zur Maximalkorrelation, *Z. Angew. Math. Mech.* **29,** 127 (1949). M **11,** 42

Rider, P. R., On the distribution of the correlation coefficient in small samples, *Biometrika* **24,** 382 (1932). Z **6,** 21

Rietz, H. L., A simple non-normal correlation surface, *Biometrika* **24,** 288 (1932). Z **7,** 72

Rietz, H. L., On the distribution of the "Student" ratio for small samples from certain non-normal populations, *Ann. Math. Statist.* **10,** 265 (1939). M **1,** 23

Romanovsky, V., Sur la loi sinusoidale limite, *Rend. Circ. Mat. Palermo* **56,** 82 (1932). Z **4,** 264

Romanovsky, V., Sur une généralisation de la loi sinusoidale limite, *Rend. Circ. Mat. Palermo* **57,** 130 (1933). Z **6,** 359

Roos, C. F., A general invariant criterion of fit for all lines and planes where all variates are subject to error, *Metron* **13,** 3 (1937). Z **16,** 172

Rosenblatt, M., Some regression problems in time series analysis, *Proc. Symp. Math. Stat. and Prob.* I, pp. 165–186 (Un. Cal. Press, Berkeley, 1956). M **18,** 948

Rosenblatt, M., On the estimation of regression coefficients of a vector-valued time series with a stationary residual, *Ann. Math. Statist.* **27,** 99 (1956). M **17,** 871

Rosner, B. S., A new scaling technique for absolute judgments, *Psychometrika* **21,** 377 (1956). M **18,** 548

♦Rothschild, C., Sur les lois de probabilité à régression linéaire et écart type lié constant, *C. R. Acad. Sci., Paris* **225,** 1117 (1947). M **9,** 599

Roy, S. N., A note on critical angles between two flats in hyperspace with certain statistical applications, *Sankhyā* **8,** 177 (1947). M **10,** 134

♦Roy, S. N., The distribution of the root-mean-square of the second type of the multiple correlation coefficient, *Science and Culture* **6,** 59 (1940). M **5,** 126

Rubin, H., On the distribution of the serial correlation coefficient, *Ann. Math. Statist.* **16,** 211 (1945). M **7,** 20

Salvemini, T., Di uno scarto trigonometrico medio, nel caso delle serie cicliche, *Atti Secondo Congresso Un. Mat. Ital. Bologna:* pp 657–671 (1940). M. **8,** 474

Salvemini, T., L'indice di cograduazione del Gini nel caso di serie statistiche con ripetizioni, *Metron* **13,** 27 (1939). Z **22,** 373

Salvemini, T., Nuovi procedimenti di calcolo degli indici di dissomiglianza e di connessione, *Statistica, Milano* **9,** 3 (1949). M **11,** 446

Samuelson, P. A., A note on alternative regressions, *Econometrica* **10,** 80 (1942). M **3,** 173

Sapogov, N. A., On the mutual disposition of the lines of regression, *Uspehi Matem. Nauk.* **9,** 187 (1954). M **16,** 270

Sapožkov, M. A., Correlational method of measurement of the coefficient of distortion of transmission, *Akustische Z.* **2,** 279 (1956). M **18,** 368

Sarmanov, O., On isogeneous correlation, *Bull. Acad. Sci. URSS Ser. Math.* **9,** 169 (1945). M **7,** 130

Sarmanov, O. V., On the rectification of a symmetric correlation, *Doklady Akad. Nauk. SSSR* **58,** 745 (1947). M **10,** 45

Sarmanov, O. V., On the rectification of asymmetrical correlation, *Doklady Akad. Nauk. SSSR* **59,** 861 (1948). M **9,** 442

Sarmanov, O. V., On the order of magnitude of a line of regression, *Doklady Akad. Nauk. SSSR* **59,** 1061 (1948). M **9,** 442

Sarmanov, O. V., On the order of magnitude of a line of regression. II, *Doklady Akad. Nauk. SSSR* **60,** 545 (1948). M **9,** 593

Sarmanov, O. V., On the rectification of correlation, *Uspehi Matem. Nauk.* **3,** 190 (1948). M **10,** 305

Sarmanoff, O. W., Sur la corrélation isogène, *C. R. Doklady Acad. Sci. URSS* **32,** 28 (1941). M **3,** 173

Sato, R., The *r* tests relating to the regression, *Ann. Inst. Statist. Math. Tokyo* **3,** 45 (1951). M **13,** 571

Sawkins, D. T., Simple regression and correlation, *J. Proc. Roy. Soc. New South Wales* **77,** 85 (1944). M **6,** 5

Sawkins, D. T., The use of cumulative graphs for estimation of means, higher moments, etc., *Metron* **3,** 33 (1938). Z **20,** 148

Scheffé, H., An inverse problem in correlation theory, *Amer. Math. Monthly* **49,** 99 (1942). M **4,** 23

von Schelling, H., *Die wirtschaftlichen Zeitreihen als Problem der Korrelationsrechnung. Mit besonderer Berücksichtigung der "lag" correlation,* Kurt Schroeder, Bonn (1931). Z **4,** 15

von Schelling, H., Korrelationsmessung auf Grund der Anordnung der Beobachtungen, *Z. Angew. Math. Mech.* **12,** 377 (1932). Z **6,** 22

Schützenberger, M. P., Valeurs caractéristiques du coefficient de corrélation par rang de Kendall dans le cas général, *C. R. Acad. Sci., Paris* **226,** 2122 (1948). M **10,** 134

♦Scott, J. F., A numerical investigation of least squares regression involving trend-reduced Markoff series, *J. Roy. Statist. Soc. Ser. B.* **17,** 105 (1955). M **17,** 505

Seares, F. H., Regression lines and the functional ralation, *Astrophys. J.* **100,** 255 (1944). M **6,** 91

Sherman, J., The use of orthogonal polynomials in curve fitting and regression analysis, *Proc. Industrial Computation Seminar* **1950**, I.B.M., N.Y. (1951). M **13,** 481

Shrivastava, M. P., Bi-variate correlation surfaces, *Science and Culture* **6,** 615 (1941). M **5,** 126

Sibagaki, W., On the theory of statistical adjustment of data. I. Conditions containing parameters, *J. Phys. Soc. Japan* **4,** 216 (1949). M **12,** 513

Simaika, J. B., Note on M. Fréchet index of correlation, *Proc. Math. Phys. Soc. Egypt* **3,** 21 (1946). M **8,** 592

Sinha, G., A note on the expression for the sample estimate of the coefficient of partial correlation, *Bull. Calcutta Math. Soc.* **41,** 159 (1949). M **11,** 259

Sitaraman, B., On correlation constants in mingled records, *Math. Student* **8,** 73 (1940). M **2,** 234

Slutsky, E., Alcune applicazioni dei coefficienti di Fourier all'analisi delle funzioni aleatorie stazionarie, *Giorn. Ist. Ital Attuari* **5,** 435 (1934). Z **10,** 312

Small, V. J., (See J. F. Scott) *J. Roy Statist. Soc. Ser. B* **17,** 105 (1955).

Smith, B. B., Note on an alternant suggested by statistical theory, *Edinburgh Math. Notes No.* **32,** 19 (1941). M **4,** 105

Smoliakow, P. T., Die Fechnersche Korrelationsformel, *Meteorol. Z.* **50,** 87 (1933). Z **6,** 267

Spoerl, C. A., The Whittaker-Henderson graduation formula A, *Trans. Actuar. Soc. Amer.* **38,** 403 (1937). Z **20,** 155

Srinivasiengar, C. N., Remarks on spurious correlation, *J. Indian Math. Soc.* **19,** 251 (1932). Z **6,** 121

Stange, K., Über das Ausgleichen einer fehlerhaften linearen Punktreihe bei korrelativer Verknüpfung der Messfehler, *Mitteilungsblatt Math. Statist.* **4,** 48 (1952). M **13,** 963

Starkey, D. M., The distribution of the multiple correlation coefficient in periodogram analysis, *Ann. Math. Statist.* **10,** 327 (1939). M **1,** 152

Steffensen, J. F., Deux problèmes du calcul des probabilités, *Ann. Inst. H. Poincaré* **3,** 319 (1933). Z **7,** 21

Steffensen, J. F., On certain measures of dependence between statistical variables, *Biometrika* **26,** 251 (1934). Z **9,** 221

Steffensen, J. F., On the coefficient of correlation for continuous distributions, *Skand. Aktuarietidskr.* **1941,** 1 (1941). M **3,** 5

Steiner, L., Zur Deutung des Quadrats des Korrelationskoeffizienten, *Meteor, Z.* **48,** 350 (1931). Z **3,** 18

Stevens, W. L., Mathematical theory of some distributions used in statistics, *Revista Fac. Ci. Univ. Coimbra* **10,** 247 (1942). M **8,** 161

Stevens, W. L., Asymptotic regression, *Biometrics* **7,** 247 (1951). M **13,** 571

Steyn, H. S., On regression properties of discrete systems of probability functions, *Indag. Math.* **19,** 119 (1957). M **18,** 771

Stouffer, S. A., A coefficient of "combined partial correlation" with an example from sociological data, *J. Amer. Statist. Assoc.* **29,** 70 (1934). Z **8,** 370

Stouffer, S. A., Evaluating the effect of inadequately measured variables in partial correlation analysis, *J. Amer. Statist. Assoc.* **31,** 348 (1936). Z **14,** 358

Strecker, G., On evaluating a coefficient of partial correlation, *Ann. Math. Statist.* **6,** 143 (1935). Z **12,** 17

♦Stuart, A. (See J. Durbin), *J. Roy. Statist. Soc. Ser. B.* **13,** 303 (1951).

Stuart, A., The estimation and comparison of strengths of association in contingency tables *Biometrika* **40,** 105 (1953). M **15,** 4

Subramanian, S., On a property of partial correlation, *J. Roy. Statist. Soc.* **98,** 129 (1935). Z **10,** 40

Swineford, F., A table for estimating the significance of the difference between correlated percentages, *Psychometrika* **13,** 23 (1948). M **9,** 363

Tate, R. F., The theory of correlation between two continuous variables when one is dichotomized, *Biometrika* **42,** 205 (1955). M **17,** 54

♦Taylor, E. K. (See R. J. Wherry), *Psychometrika*, **11,** 155 (1946).

Teatini, U., Degli indici di correlazione tra caratteri ciclici, *Statistica, Milano* **10,** 46 (1950). M **12,** 38

Teghem, J., Sur la régression polynomiale, dans le cas où les valeurs de la variable indépendante sont en progression arithmétique lacunaire, *Bull. Inst. Internat. Statist.* **24,** 109 (1954). M **16,** 1132

Teodorescu, C. C., La corrélation statistique, *Bull. Sci. École Polytechn. Timisoara* **4,** 202 (1932). Z **6,** 267

Teodoresco, C. C., Note sur la corrélation, *Mathematica, Cluj* **9,** 294 (1935). Z **13,** 314

Theil, H., A rank-invariant method of linear and polynomial regression analysis, *Indagationes Math.* **12,** 85 (1950). M **12,** 117

Theil, H., A rank-invariant method of linear and polynomial regression analysis. II, *Indagationes Math.* **12,** 173 (1950). M **12,** 117

Theil, H., A rank-invariant method of linear and polynomial regression analysis. III, *Indagationes Math.* **12,** 467 (1950). M **12,** 725

Theil, H., Distribution-free methods in the regression analysis of two variables, *Statistica, Rijswijk* **5,** 97 (1951). M **13,** 571

Thurstone, L. L., Graphical method of factoring the correlation matrix, *Proc. Nat. Acad. Sci. U.S.A.* **30,** 129 (1944). M **6,** 6

Thurstone, L. L., A multiple group method of factoring the correlation matrix, *Psychometrika* **10,** 73 (1945). M **7,** 20

Tintner, G., The Variate Difference Method, *Monograph 5, Cowles Comm.* (Principia Press, Bloomington, Ind. 1940). M **1,** 250

Tintner, G., An application of the variate difference method to multiple regression, *Econometrica* **12,** 97 (1944). M **6,** 93

Tintner, G., Die Anwendung der Variate Difference Methode auf die Probleme der gewogenen Regression und der Multikollinearität, *Mitteilungsblatt Math. Statist.* **4,** 159 (1952). M **14,** 297

Tocher, K. D., On the concurrence of a set of regression lines, *Biometrika* **39,** 109 (1952). M **13,** 963

Treloar, A. E., *Correlation analysis* (Burgess Publishing Co., Minneapolis, Minn. 1942). M **4,** 220

Třiska, K., How to compute the standard error of a correlation with ρ, *Aktuár. Vědy* **4,** 134 (1934). Z **9,** 222

Tsao, F., General solution of the analysis of variance and covariance in the case of unequal or disproportionate numbers of observations in the subclasses, *Psychometrika* **11,** 107 (1946). M **8,** 42

Tschuprow, A. A., *Principles of the mathematical theory of correlation* (W. Hodge Co., London, 1939). M **1,** 151

Tucker, L. R., The determination of successive principal components without computation of tables of residual correlation coefficients, *Psychometrika* **9,** 149 (1944). M **6,** 51

♦Tukey, J. W., (See P. S. Olmstead) *Ann. Math. Statist.* **18,** 495 (1947).

Udny Yule, G., On a parallelism between differential coefficients and regression coefficients, *J. Roy. Statist. Soc.* **99,** 770 (1936). Z **15,** 118

Vacek, M., Sur la loi de Polya régissant les faits corrélatifs, *Aktuár. Vědy* **3,** 18 and 49 (1932). Z **6,** 69

Vajda, S., The algebraic analysis of contingency tables, *J. Roy. Statist. Soc.* **106,** 333 (1943). M **6,** 162

Vajda, S., Relations between variously defined effects and interactions in analysis of variance, *Ann. Math. Statist.* **22,** 2[illegible]3 (1951). M **12,** 843

Van Boven, A., A modified Aitken pivotal condensation method for partial regression and multiple correlation, *Psychometrika* **12,** 127 (1947). M **9,** 47

♦van der Laan, E., Evaluation of a regression by means of orthogonal polynomials, *Statistica, Leiden* **1,** 219 (1947). M **11,** 259

Vaswani, S. P., A pitfall in correlation theory, *Nature* **160,** 405 (1947). M **9,** 294

Vaswani, S., Assumptions underlying the use of the tetrachoric correlation coefficient, *Sankhyā* **10,** 269 (1950). M **12,** 429

♦Venkatachari, S., (See G. V. Krishnaswami), *J. Annamalai Univ.* **3,** 189 (1934).

Vincze, I., Sur la détermination du coefficient de régression, *Magyar Tud. Akad. Alkalm. Mat. Int. Közl.* **1,** 215 (1953). M **15,** 141

♦Votaw, D. F., Compound symmetry tests in the multivariate analysis of medical experiments, *Biometrics* **6,** 259 (1950). M **12,** 271

Wald, A., A note on regression analysis, *Ann. Math. Statist.* **18,** 586 (1947). M **9,** 364

Wallis, W. A., The correlation ratio for ranked data, *J. Amer. Statist. Assoc.* **34,** 533 (1939). Z **23,** 242

Wallis, W. A., Tolerance intervals for linear regression, *Proc. Symp. Math. Stat. Prob., pp. 43–51,* (Univ. Cal. Press, Berkeley 1951). M **13,** 481

Wasastjerna, J. A., The configurational partition functions for binary solid solutions and the equilibrium degrees of local and long range order, *Soc. Sci. Fenn. Comment. Phys. Math.* **14,** 8 pp. (1948). M **13,** 714

♦Watson, G. S., (See J. Durbin), *Biometrika* **37,** 409 (1950).

Watson, G. S., A note on the circular multivariate distribution, *Biometrika* **43,** 467 (1956). M **18,** 522

Watson, G. S., Serial correlation in regression analysis I, *Biometrika* **42,** 327 (1955). M **17,** 382

Waugh, F. V., A simplified method of determining multiple regression constants, *J. Amer. Statist. Assoc.* **30,** 694 (1935). Z **15,** 362

Waugh, F. V., The analysis of regression in subsets of variables, *J. Amer. Statist. Assoc.* **31,** 729 (1936). Z **15,** 362

Waugh, F. V., The computation of partial correlation coefficients, *J. Amer. Statist. Assoc.* **41,** 543 (1946). M **8,** 282

Wei, D., Necessary and sufficient conditions that regression systems of sums with elements in common be linear, *Nat. Math. Mag.* **17,** 151 (1943). M **4,** 164

Weichelt, J. A., A first-order method for estimating correlation coefficients, *Psychometrika* **11,** 215 (1946). M **8,** 282

Welch, B. L., Some problems in the analysis of regression among k samples of two variables, *Biometrika* **27,** 145 (1935). Z **11,** 220

Welch, B. L., On the comparison of several mean values: an alternative approach, *Biometrika* **38,** 330 (1951). M **13,** 762

♦Welker, E. L. (See D. M. Hall), *Psychometrika* **10,** 93 (1945).

Wells, E. H., The mechanical side of mechanicographic graduation, *Trans. Actuar. Soc. Amer.* **38,** 384 (1937). Z **20,** 155

Wherry, R. J., A new formula for predicting the shrinkage of the coefficient of multiple correlation, *Ann. Math. Statist.* **2,** 440 (1931). Z **4,** 265

Wherry, R. J., Multiple bi-serial and multiple point bi-serial correlation, *Psychometrika* **12,** 189 (1947). M **9,** 47

♦Wherry, R. J., The relation of multiserial eta to other measures of correlation, *Psychometrika* **11,** 155 (1946). M **8,** 41

Whitfield, J. W., Rank correlation between two variables, one of which is ranked, the other dichotomous, *Biometrika* **34,** 292 (1947). M **9,** 453

Whittle, P., The statistical analysis of a seiche record, *J. Marine Res.* **13,** 76 (1954). M **17,** 1219

Wicksell, S. D., On correlation functions of type III, *Biometrika* **25,** 121 (1933). Z **7,** 124

Wicksell, S. D., Analytical theory of regression, *Lunds Univ. Arsskr. N. F.* **30,** 1 (1934). Z **9,** 266

Wilder, M. A., Correlation coefficients and transformation of axes, *Amer. Math. Monthly* **38,** 64 (1931). Z **1,** 151

♦Wilks, S. S. (See H. Gulliksen), *Psychometrika* **15,** 91 (1950).

Wilks, S. S., Moments and distributions of estimates of population parameters from fragmentary samples, *Ann. Math. Statist.* **3,** 163 (1932). Z **5,** 73

Willers, Fr. A., Korrelation zwischen drei Veränderlichen, *Skand. Aktuarietidskr* **14,** 158 (1931). Z **3,** 164

Williams, E. J., Tests of significance for concurrent regression lines, *Biometrika* **40,** 297 (1953). M **17,** 641

Williams, E. J., Use of scores for the analysis of association in contingency tables, *Biometrika* **39,** 274 (1952). M **17,** 641

Wilson, E. B., Correlation and association, *J. Amer. Statist. Assoc.* **26,** 250 (1931). Z **2,** 282

Wilson, E. B., The controlled experiment and the four-fold table, *Science* **93,** 557 (1941). M **4,** 26

♦Wilson, E. B., The resolution of six tests into three general factors, *Proc. Nat. Acad. Sci. U. S. A.* **25,** 73 (1939). Z **21,** 147

♦Wilson, E. B., Contingency tables, *Proc. Nat. Acad. Sci. U. S. A.* **28,** 378 (1942). M **4,** 105

♦Wilson, E. B., The association of three attributes, *Proc. Nat. Acad. Sci. U. S. A.* **28,** 384 (1942). M **4,** 106

Wise, J., Regression analysis of relationships between autocorrelated time series, *J. Roy. Statist. Soc. Ser. B* **18,** 240 (1956). M **18,** 955

Wishart, J., The mean and second moment coefficient of the multiple correlation coefficient in samples from a normal population, *Biometrika* **22,** 353 (1931). Z **2,** 41

Wishart, J., A note on the distribution of the correlation ratio, *Biometrika* **24,** 441 (1932). Z **6,** 22

Wishart, J., Tests of significance in the simple regression problem, *J. Inst. Actuaries Students' Soc.* **8,** 38 (1948). M **10,** 50

Wishart, J., Test of homogeneity of regression coefficients, and its application in the analysis of variance, *Colloq. Int. Centre Nat. Reche. Sci.* **13,** pp. 93–99, Paris (1949). M **11,** 445

Wold, H., Sheppard's correction formulae in several variables, *Skand. Aktuarietidskr.* **17,** 248 (1934). Z **10,** 214

Wold, H., On quantitative statistical analysis, *Skand. Aktuarietidskr.* **19,** 281 (1936). Z **15,** 220

Wold, H., A theorem on regression coefficients obtained from successively extended sets of variables, *Skand. Aktuarietidskr.* **28,** 181 (1945). M **7,** 317

Wold, H., A comment on spurious correlation, *Försäkringsmatematiska Studier Lundberg*, pp. 278–285, Stockholm (1946). M **8,** 393

Wold, H., Über gleichmässige Interkorrelation, *Mitteilungsblatt Math. Statist.* **4,** 163 (1952) M **14,** 297

Wolfowitz, J., Consistent estimators of the parameters of a linear structural relation, *Skand. Aktuarietidskr.* **35,** 132 (1953). M **14,** 776

Wong, Y. K., On the elimination of variables in multiple correlation, *J. Amer. Statist. Assoc.* **32,** 357 (1937). Z **16,** 314

Woodbury, M. A., Rank correlation when there are equal variates, *Ann. Math. Statist.* **11,** 358 (1940). M **2,** 110

Woolley, E. B., The method of minimized areas as a basis for correlation analysis, *Econometrica* **9,** 38 (1941). M **2,** 235

♦Worcester, J., (See E. B. Wilson), *Proc. Nat. Acad. Sci. U.S.A.* **28,** 378 (1942).

♦Worcester, J., (See E. B. Wilson), *Proc. Nat. Acad. Sci., U.S.A.* **25,** 73 (1939).

♦Worcester, J., (See E. B. Wilson), *Proc. Nat. Acad. Sci. U.S.A.* **28,** 384 (1942).

Wright, S., The method of path coefficients, *Ann. Math. Statist.* **5,** 161 (1934). Z **10,** 313

Yates, F., Tests of significance of the differences between regression coefficients derived from two sets of correlated variates, *Proc. Roy. Soc. Edinburgh* **59,** 184 (1939). M **1,** 23

Young, G., Maximum likelihood estimation and factor analysis, *Psychometrika* **6,** 49 (1941). M **2,** 235

Yuan, P. T., On the logarithmic frequency distribution and the semi-logarithmic correlation surface, *Ann. Math. Statist.* **4,** 30 (1933). Z **6,** 359

Zimmerman, W. S., A simple graphical method for orthogonal rotation of axes, *Psychometrika* **11,** 51 (1946). M **7,** 338

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Selected Bibliography of Statistical Literature, 1930 to 1957: II. Time Series

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This is the second in a series of bibliographies dealing with various specific subjects in the field of statistics. References and titles of important contributions to the study of time series have been taken from a wide variety of technical journals published in the many languages and countries which have been actively engaged in statistical analysis.

If this series of subject-classified bibliographies can assist the overburdened statistician to keep abreast of the great abundance of literature in his field, our aim has been accomplished. Although the coverage of technical journals is worldwide, each bibliography is still far from complete. The books and articles considered by two prominent reviewing journals to be of sufficient statistical importance to include in their regular abstracting services form the source material for these classified bibliographies.

This particular subject classification on *Time Series* follows the earlier and closely related one on *Correlation and Regression Theory*.[1] The titles and references are extracted from a card file made up of abstracts taken from *Zentralblatt für Mathematik* for the years 1930 to 1939, and from *Mathematical Reviews* from 1940 through 1957. The file of abstracts is maintained on a current basis in the NBS Statistical Engineering Laboratory; the bibliographies extracted from it are necessarily delayed due to loss of time in processing, editing, and publishing.

The following information comes directly from the abstracts:

Author: The author's surname is followed by initials only. In the case of complicated surnames, the first capitalized word given in the reviewing journal is used. When the paper bears the names of more than one author, the journal reference appears with each author's name, but the title of the paper is given with the senior author only. Multiple authorship is denoted by the symbol ♦ preceding the surname.

Title: Given exactly as in the reviewing journal. Titles of separately bound publications (books, reports, etc.) are italicized, and are followed by the publisher.

Reference to literature: In many cases the abbreviations of journal names have been necessarily severe and unconventional. The name of the journal in italics and the number of the volume in boldface, are followed by the initial page number.

Date of publication: The next figure shows the date in parentheses in which the article or book itself appeared.

M (for *Mathematical Reviews*) and *Z* (for *Zentralblatt für Mathematik*) are followed by the volume number and the page number of the reviewing journal in which the abstract appears.

Anderson, R. L. & T. W., Distribution of the circular serial correlation coefficient for residuals from a fitted Fourier series, *Ann. Math. Statist.* **21,** 59 (1950). M **11,** 530

Anderson, R. L., The problem of autocorrelation in regression analysis, *J. Amer. Statist. Assoc.* **49,** 113 (1954). M **15,** 728

♦Anderson, T. W., Statistical inference about Markov chains, *Ann. Math. Statist.* **28,** 89 (1957). M **18,** 944

Barricelli, N. A., Les plus grands et les plus petits maxima ou minima annuels d'une variable climatique, *Arch. Math. Naturvid.* **46,** 155 (1943). M **7,** 22

Bartlett, M. S., Periodogram analysis and continuous spectra, *Biometrika* **37,** 1 (1950). M **12,** 35

♦Bartlett, M. S., Extensions of Quenouille's test for autoregressive schemes, *J. Roy. Statist. Soc.* (*Ser. B*) **12,** 108 (1950). M **12,** 512

♦Bartlett, M. S., On the efficiency of procedures for smoothing periodograms from time series with continuous spectra, *Biometrika* **42,** 143 (1955). M **16,** 1040

♦Bartlett, M. S., Goodness of fit tests for simultaneous autoregressive series, *J. Roy. Statist. Soc.* (*Ser. B*) **15,** 107 (1953). M **15,** 333

Bartlett, M. S., Problèmes de l'analyse spectrale des séries temporelles stationnaires, *Publ. Inst. Statist. Univ. Paris* **3,** 119 (1954). M **16,** 731

Bartlett, M. S., On the theoretical specification and sampling properties of autocorrelated time-series, *Suppl. J. Roy. Statist. Soc.* **8,** 27 (1946). M **8,** 283

[1] See p. 55.

Bartlett, M. S., Corrigenda to the paper "On the theoretical specification and sampling properties of autocorrelated time-series, *J. Roy. Statist. Soc.* (*Ser. B*) **10,** (1948). M **10,** 724

Bates, G. E., Joint distributions of time intervals for the occurrence of successive accidents in a generalized Pólya scheme, *Ann. Math. Statist.* **26,** 705 (1955). M **17,** 872

♦Bayley, G. V., The "effective" number of independent observations in an autocorrelated time series, *Suppl. J. Roy. Statist. Soc.* **8,** 184 (1946). M **8,** 591

♦Bentzel, R., On statistical demand analysis from the viewpoint of simultaneous equations, *Skand. Aktuarietidskr.* **29,** 95 (1946). M **8,** 216

Berger, E. R., Harmonische Analyse diskreter Zahlenreihen, *Z. Angew. Math. Mech.* **22,** 269 (1942). M **5,** 51

♦Birdsall, T. G., (See D. Middleton), *J. Appl. Phys.* **25,** 128 (1954).

Blake, A., Criteria for the reality of apparent periodicities and other regularities, *Publ. Bur. Central Séismologique Internat.* (*Ser. A*) No 16, (1946). M **9,** 603

Blanc-Lapierre, A., Sur quelques modèles statistiques suggérés par l'étude de l'effet de scintillation, *Publ. Inst. Statist. Univ. Paris* **2,** 3 (1953). M **15,** 811

Blassel, P., Erreur due à une durée d'intégration finie dans la détermination des fonctions d'autocorrélation, *Ann. Télécommun.* **8,** 406 (1953). M **15,** 454

Bochner, S., Fourier transforms of time series, *Proc. Nat. Acad. Sci. U. S. A.* **39,** 302 (1953). M **14,** 1101

Brelot, Sur l'influence des erreurs de mesure en statistique et biométrie, *Extrait du Bull. Trav. Publ. Stat. Aquicult. et Pêche Castiglione,* 32 pp. (1937). Z **19,** 130

Brown, G. W., Notes on the solution of linear systems involving inequalities, *Proc. Symp. Digital Calc. Mach.* (Harvard Univ. Press. 1951). M **13,** 261

♦Bryan, J. G., (See G. P. Wadsworth), *Geophysics* **18,** 539 (1953).

Castellano, V., Contributo allo studio delle serie cicliche, *Statistica, Bologna* **15,** 23 (1955). M **16,** 1134

Champernowne, D. G., Sampling theory applied to autoregressive sequences, *J. Roy. Statist. Soc.* (*Ser. B*) **10,** 204 (1948). M **10,** 724

Chandler, K. N., The distribution and frequency of record values, *J. Roy. Statist. Soc.* (*Ser. B*) **14,** 220 (1952). M **14,** 778

♦Cox, D. R., The superposition of several strictly periodic sequences of events, *Biometrika* **40,** 1 (1953). M **14,** 1105

Cruickshank, A. J. O., A note on time series and the use of jump functions in approximate analysis, *Proc. Inst. Elec. Engrs.* (*C*) **102,** 81 (1955). M **16,** 1134

Curtiss, J. H., Acceptance sampling by variables, with special reference to the case in which quality is measured by average or dispersion, *J. Research Nat. Bur. Standards* **39,** 271 (1947). M **9,** 366

Daniels, H. E., The theory of position finding, *J. Roy. Statist. Soc.* (*Ser. B*) **13,** 186 (1951). M **13,** 963

Daniels, H. E., The approximate distribution of serial correlation coefficients, *Biometrika* **43,** 169 (1956). M **18,** 79

Das, A. C., On the estimation of parameters in a recursive system, *Sankhyā* **11,** 273 (1951). M **14,** 66

♦Davenport, W. B., Statistical errors in measurements on random time functions, *J. Appl. Phys.* **23,** 377 (1952). M **13,** 763

Davis, H. T., The Analysis of Economic Time Series, *Monograph, No. 6, Cowles Commission,* Principia Press, Bloomington, Ind.(1941). M **3,** 176

Davis, H. T., The statistics of time series, *Math. Monographs, Vol. 1,* pp. 45–85, Northwestern Univ. Evanston, Ill. (1941). M **3,** 175

♦Diananda, P. H., (See M.S. Bartlett.) *J. Roy. Statist. Soc.* (*Ser. B*) **12,** 108 (1950).

Dieulefait, C. E., On Slutsky's sinusoidal limit law, derived from a new sequence of random variables, *An. Soc. Ci. Argentina* **134,** 257 (1942). M **5,** 207

Dodd, E. L., The length of the cycles which result from the graduation of chance elements, *Ann. Math. Statist.* **10,** 254 (1939). M **1,** 23

Dodd, E. L., The problem of assigning a length to the cycle to be found in a simple moving average and in a double moving average of chance data, *Econometrica* **9,** 25 (1941). M **2,** 237

Dodd, E. L., The cyclic effects of linear graduations persisting in the differences of the graduated values, *Ann. Math. Statist.* **12,** 127 (1941). M **3,** 10

Domb, C., The statistics of correlated events, *Philos. Mag.* **41,** 969 (1950). M **12,** 426

Doob, J. L., The elementary Gaussian processes, *Ann. Math. Statist.* **15,** 229 (1944). M **6,** 89

Doob, J. L., Time series and harmonic analysis, *Proc. Symp. Math. Stat. Prob,* pp. 303–343, Univ. Cal. Press, Berkeley (1949). M **10,** 385

♦Dvoretzky, A., Sequential decision problems for processes with continuous-time parameter. Testing hypotheses, *Ann. Math. Statist.* **24,** 254 (1953). M **14,** 997

♦Eisenhart, C., *Selected techniques of statistical analysis for scientific and industrial research and production and management engineering* (McGraw-Hill Book Co., Inc., New York, 1947). M **9,** 365

Faesi, M., Über die Glättung statistischer Verteilungsreihen, *Mitt. Verein. Schweiz. Versich. Math.* **40,** 61 (1940). M **3,** 10

Feather, N., On the statistics of random distributions of paired events, with applications to the results obtained in the use of the interval selector with particle counters, *Proc. Cambridge Philos. Soc.* **39,** 84 (1943). M **4,** 223

♦Féron, R., (See C. Fourgeaud), *Publ. Inst. Statist., Univ. Paris* **1,** 21 (1952).

♦Foster, F. G., Distribution free tests in time-series based on the breaking of records, *J. Roy. Statist. Soc. (Ser. B)* **16,** 1 (1954). M **16,** 385

♦Foster, F. G., A sampling experiment on the powers of the records tests for trend in a time series, *J. Roy. Statist. Soc. (Ser. B)* **17,** 115 (1955). M **17,** 382

♦Fourgeaud, C., Quelques remarques sur l'estimation des variations saisonnières, *Publ. Inst. Statist. Univ. Paris* **1,** 21 (1952). M **15,** 811

Fränz, K., Relations between signal and spectrum, *Revista Unión Mat. Argentina* **14,** 140 (1950). M **11,** 660

Galletti, R., A note on graphs, *J. Madras Univ. (Sect. B)* **15,** 19 (1943). M **7,** 87

Geary, R. C., Studies in relations between economic time series, *J. Roy. Statist. Soc. (Ser. B)* **10,** 140 (1948). M **10,** 553

Geary, R. C., Determination of linear relations between systematic parts of variables with errors of observation the variances of which are unknown, *Econometrica* **17,** 30 (1949). M **10,** 465

Gebelein, H., Maximalkorrelation und Korrelationsspektrum, *Z. Angew. Math. Mech.* **32,** 9 (1952). M **13,** 669

Geisser, S., The modified mean square successive difference and related statistics, *Ann. Math. Statist.* **27,** 819 (1956). M **18,** 423

Ghidoli, G. B., Generalizzazione della legge sinusoidale, *Statistica, Milano* **9,** 72 (1949). M **11,** 672

Ghosh, B., A multi-stage stochastic model for some natural fields, *Calcutta Statist. Assoc. Bull.* **3,** 21 (1950). M **12,** 347

Ghurye, S. G., A method of estimating the parameters of an autoregressive time-series, *Biometrika* **37,** 173 (1950). M **12,** 118

Gibson, G. D., A rapid method for ascertaining serial lag correlation, *Biometrika* **37,** 288 (1950). M **12,** 430

Gleissberg, W., Ein Kriterium für die Realität zyklischer Variationen, *Rev. Fac. Sci. Univ. Istanbul (A)* **10,** 36 (1945). M **8,** 40

Glock, W. S., A rapid method of correlation for continuous time series, *Amer. J. Sci.* **240,** 437 (1942). M **4,** 27

♦Goodman, L. A., (See T. W. Anderson), *Ann. Math. Statist.* **28,** 89 (1957).

Goodman, L. A., A further note on "Finite Markov processes in psychology," *Psychometrika* **18,** 245 (1953). M **15,** 333

Grant, A. M., Some properties of runs in smoothed random series, *Biometrika* **39,** 198 (1952). M **13,** 963

Grenander, U., On the estimation of regression coefficients in the case of an autocorrelated disturbance, *Ann. Math. Statist.* **25,** 252 (1954). M **15,** 973

♦Grenander, U., On spectral analysis of stationary time series, *Proc. Nat. Acad. Sci. U.S.A.* **38,** 519 (1952). M **14,** 61

♦Grenander, U., *Statistical analysis of stationary time series,* (John Wiley & Sons, N.Y., 1957). M **18,** 959

♦Grenander, U., Some problems in estimating the spectrum of a time series, *Proc. Symp. Math. Stat. Prob,* **1** pp. 77–93, U. Calif. Pres., Berkeley (1956). M **18,** 946

♦Grenander, U., Statistical spectral analysis of time series arising from stationary stochastic processes, *Ann. Math. Statist.* **24,** 537 (1953). M **15,** 448

♦Grenander, U., Comments on statistical spectral analysis, *Skand. Aktuarietidskr.* **36,** 182 (1953). M **15,** 728

♦Grenander, U., An extension of a theorem of G. Szegö and its application to the study of stochastic processes, *Trans. Amer. Math. Soc.* **76,** 112 (1954). M **15,** 448

♦Grenander, U., Regression analysis of time series with stationary residuals, *Proc. Nat. Acad. Sci. U.S.A.* **40,** 812 (1954). M **15,** 974

Gumbel, E. J., Probability-interpretation of the observed return-periods of floods, *Trans. Amer. Geophys. Union* **1941,** 836 (1941). M **4,** 29

Gumbel, E. J., La durée de retour, *La Météorologie* **1942,** 71 (1943). M **7,** 319

Gumbel, E. J., Simplified plotting of statistical observations, *Trans. Amer. Geophys. Union* **26,** 69 (1945). M **7,** 133

Gurland, J., An example of autocorrelated disturbances in linear regression, *Econometrica* **22,** 218 (1954). M **15,** 887

Hafstad, L. R., On the Bartels technique for time-series analysis, and its relation to the analysis of variance, *J. Amer. Statist. Assoc.* **35,** 347 (1940). M **1,** 349

Hald, A., The Decomposition of a Series of Observations Composed of a Trend, a Periodic Movement and a Stochastic Variable, *Thesis,* Univ. of Copenhagen (1948). M **9,** 363

♦Hammersley, J. M., (See G. V. Bayley), *Suppl. J. Roy. Statist. Soc.* **8,** 184 (1946).

Hannan, E. J., The asymptotic powers of certain tests based on multiple correlations, *J. Roy. Statist. Soc.* **18,** 227 (1956). M **18,** 682

Hannan, E. J., An exact test for correlation between time series, *Biometrika* **42,** 316 (1955). M **17,** 381

Hanner, O., Deterministic and non-deterministic stationary random processes, *Ark. Mat.* **1,** 161 (1950). M **11,** 606

Hartley, H. O., Second order autoregressive schemes with time-trending coefficients, *J. Roy. Statist. Soc. (Ser. B)* **14,** 229 (1952). M **14,** 778

♦Hastay, M. W., (See C. Eisenhart), McGraw-Hill Book Co., Inc., New York (1947).

Hoel, P. G., Confidence bands for polynomial curves, *Ann. Math. Statist.* **25,** 534 (1954). M **16,** 155

♦Hurley, P. M., (See G. P. Wadsworth), *Geophysics* **18,** 539 (1953).

♦Huruya, S., (See T. Kitagawa), *Mem. Fac. Sci. Kyūsyū Imp. Univ.* (*A*) **2,** 151 (1942).

Hurwicz, L., Stochastic models of economic fluctuations, *Econometrica* **12,** 114 (1944). M **6,** 11

♦Hurwitz, H., Statistical analysis of certain types of random functions, *Ann. Math. Statist.* **15,** 173 (1944). M **6,** 89

Huttly, N. A., The fitting of regression curves with auto-correlated data, *Biometrika* **43,** 468 (1956). M **18,** 522

♦Ikeda, N., (See T. Seguchi), *Mem. Fac. Sci. Kyūsyū Univ.* (*A*) **8,** 187 (1954).

Insolera, F., Sulla durata massima della vita umana, *Giorn. Mat. Finanz.* **II**, 1 (1938). Z **20,** 47

Itô, K., On the distribution of population by ages, *Proc. Phys. Math. Soc. Jap. III*, **20**, 908 (1938). Z **20,** 47

Jenkins, G. M., Tests of hypotheses in the linear autoregressive model. II. Null distributions for higher order schemes: non-null distributions, *Biometrika* **43,** 186 (1956). M **18,** 79

Jenkins, G. M., An angular transformation for the serial correlation coefficient, *Biometrika* **41**, 261 (1954). M **15,** 974

♦Johnson, R. A., (See W. B., Davenport), *J. Appl. Phys.* **23,** 377 (1952).

Johnson, N. L., Tests of significance in the variate difference method, *Biometrika* **35,** 206 (1948). M **10,** 51

Johnson, N. L., Alternative systems in the analysis of variance, *Biometrika* **35,** 80(1948). M **10,** 313

Johnson, N. L., Time series, *J. Inst. Actuaries Students' Soc.* **10,** 13 (1950). M **12,** 430

Johnston, J., A revised test for systematic oscillation, *J. Roy. Statist. Soc.* (*Ser B*) **16,** 292 (1954). M **17,** 57

Jowett, G. H., Sampling properties of local statistics in stationary stochastic series, *Biometrika* **42,** 160 (1955). M **16,** 1134

♦Kac, M., (See H. Hurwitz), *Ann. Math. Statist.* **15,** 173 (1944).

♦Kannan, M. B., (See A. Matthai), *Sankhyā* **11,** 218 (1951).

Kanô, S., On the filter problem of a stationary stochastic process, *Bull. Math. Statist.* **5,** 47 (1953). M **15,** 638

Karhunen, K., Über die Struktur stationärer zufälliger Funktionen, *Ark. Mat.* **1,** 141 (1950). M **11,** 607

♦Kavanagh, A. J., (See O. W. Richards), *Growth* **1,** 217 (1937).

Kawata, T., On Wiener's prediction theory, *Rep. Statist. Appl. Res. Union. Jap. Sci. Eng.* **2,** 120 (1953). M **15,** 143

Kelley, T. L., The evidence for periodicity in short time series, *J. Amer. Statist. Assoc.* **38,** 319 (1943). M **5,** 51

Kendall, M. G., Tables of autoregressive series, *Biometrika* **36,** 267 (1949). M **11,** 449

Kendall, M. G., Note on bias in the estimation of autocorrelation, *Biometrika* **41,** 403 (1954). M **16,** 385

Kendall, M. G., The effect of the elimination of trend on oscillations in time-series, *J. Roy. Statist. Soc.* **104,** 43 (1941). M **3,** 10

Kendall, M. G., On autoregressive time series, *Biometrika* **33,** 105 (1944). M **6,** 163

Kendall, M. G., *Contributions to the study of oscillatory time-series* (Cambridge Univ. Press or Macmillan Co., N.Y., 1946). M **8,** 471

Kendall, M. G., *The advanced theory of statistics vol. II* (Charles Griffin Co., London, W.C. 2, 1946). M **8,** 473

Kendall, M. G., The estimation of parameters in linear autoregressive time series, *Econometrica* **17** *Suppl*, 44 (1949). M **12,** 512

Kendall, M. G., Regression, structure and functional relationship. II, *Biometrika* **39,** 96 (1952). M **14,** 66

♦Kiefer, J., (See A. Dvoretzky), *Ann. Math. Statist.* **24,** 254 (1953).

Kimball, B. F., Limited type of primary probability distribution applied to annual maximum flood flows, *Ann. Math. Statist.* **13,** 318 (1942). M **4,** 29

Kitagawa, T., Errata: Some stochastic considerations upon empirical functions of various types, *Bull. Math. Statist.* **6,** 49 (1955). M **17,** 870

Kitagawa, T., Some stochastic considerations upon empirical functions of various types, *Bull. Math. Statist.* **5,** 19 (1953). M **15,** 886

♦Kitagawa, T., The probabilistic analysis of the time-series of rare events, *Mem. Fac. Sci. Kyūsyū Imp. Univ.* (*A*) **2,** 151 (1942). M **9,** 151

♦Kivéliovitch, M., Quelques nouveaux tests pour l'étude des séries chronologiques *II*, *J. Sci. Météorol.* **6,** 151 (1954). M **17,** 1221

♦Kivéliovitch, M., Quelques nouveaux tests pour l'étude des séries chronologiques, *J. Sci. Météorol.* **6,** 73 (1954). M **16,** 942

♦Kivéliovitch, M., Étude statistique des séries chronologiques, *J. Sci. Météorol.* **5,** 21, 75, 89, 129 (1953). M **16,** 941

Kolmogoroff, A. N., Über das logarithmisch normale Verteilungsgesetz der Dimensionen der Teilchen bei Zerstückelung, *C. R. Acad. Sci. URSS* **31,** 99 (1941). M **3,** 4

Kolmogoroff, A. N., Stationary sequences in Hilbert's space *Bolletin Moskovskogo Gosudar. Univ. Matematika* **2,** 40 pp. (1941). M **5,** 101

Kolscher, M., Harmonische Analyse einer gestörten Schwingung nach Amplitudenbegrenzung, *Arch. Elektr. Übertragung* **5,** 293 (1951). M **13,** 167

Koopmans, T., The degree of damping in business cycles, *Econometrica* **8,** 79 (1940). M **1,** 155

Koopmans, T., Statistical Inference in Dynamic Economic Models, *Cowles Commission Monograph No. 10.* J. Wiley, N.Y. (1950). M **12,** 431

Kosambi, D. D., Correlation and time series, *Current Sci.* **10,** 372 (1941). M **3,** 175

Kosulajeff, P. A., Sur les problèmes d'interpolation et d'extrapolation des suites stationnaires, *C. R. Acad. Sci. URSS* **30,** 13 (1941). M **3,** 4

Koyck, L. M., Long-term foreign trade elasticities, *Metroecon.* **5,** 61 (1953). M **18,** 244

Kudô, A., Note on the estimation of the mean value of the stochastic process, *Bull. Math. Statist.* **5,** 53 (1953). M **15,** 142

Lampard, D. G., A new method of determining correlation functions of stationary time series, *Proc. Inst. Elec. Engrs. (C)* **102,** 35 (1955). M **16,** 1134

Laurent-Duhamel, M-J., Étude statistique de contours craniens considérés comme courbes aléatoires, *Publ. Inst. Statist. Univ. Paris* **2,** 27 (1953). M **15,** 887

♦Levene, H., The covariance matrix of runs up and down, *Ann. Math. Statist.* **15,** 58 (1944). M **5,** 208

Levinson, N., The Wiener RMS (root mean square) error criterion in filter design and prediction, *J. Math. Phys. Mass. Inst. Tech.* **25,** 261 (1947). M **8,** 391

Linder, A., Statistische Methoden für Naturwissenschafter, Mediziner und Ingenieure, *Monograph,* Verlag Birkhäuser, Basel, (1945). M **7,** 316

♦Lomnicki, Z. A., Some applications of zero-one processes, *J. Roy. Statist. Soc. (Ser. B)* **17,** 243 (1955). M **17,** 984

Luvsanceren, Š., Maximum likelihood estimates and confidence regions for unknown parameters of a stationary Gaussian process of Markov type, *Doklady Akad. Nauk. SSSR* **98,** 723 (1954). M **16,** 385

Mazelsky, B., Extension of power spectral methods of generalized harmonic analysis to determine non-Gaussian probability functions of random input disturbances and output responses of linear systems, *J. Aeronaut. Sci.* **21,** 145 (1954). M **15,** 545

Mandelbrot, B., On recurrent noise limiting coding, *Proc. Sym. Inform. Networks,* pp. 205–221 Polytech. Institute, Brooklyn, N.Y. (1955). M **17,** 170

Mann, H. B., A theory of estimation for the fundamental random process and the Ornstein Uhlenbeck process, *Sankhyā* **13,** 325 (1954). M **16,** 154

Mann, H. B., On the efficiency of the least square estimates of parameters in the Ornstein Uhlenbeck process, *Sankhyā* **13,** 351 (1954). M **16,** 154

♦Mann, H. B., On the statistical treatment of linear stochastic difference equations, *Econometrica* **11,** 173 (1943). M **5,** 129

♦Marriott, F. H. C., Bias in the estimation of autocorrelations, *Biometrika* **41,** 390 (1954). M **16,** 385

Marseguerra, E., Su alcuni valori caratteristici delle serie cicliche trattate con metodo continuo, *Metron* **15,** 71 (1949). M **11,** 530

♦Matthai, A., The applicability of large sample tests for moving average and autoregressive schemes to series of short length—an experimental study, Part 1: Moving averages, *Sankhyā* **11,** 218 (1951). M **14,** 391

Mattila, S., Some tests based on moving average operations on time series, *Ann. Acad. Sci. Fenn. (Ser. A)* 226, 12 pp. (1956). M **18,** 522

Mayot, M., Sur la détermination statistique des composantes cycliques d'un phénomene: application aux étoiles variables, *C. R. Acad. Sci. Paris* **223,** 125 (1946). M **8,** 40

♦Medhi, J., (See M. S. Bartlett), *Biometrika* **42,** 143 (1955).

Middleton, D., Statistical criteria for the detection of pulsed carriers in noise. II, *J. Appl. Phys.* **24,** 379 (1953). M **14,** 1105

Middleton, D., Statistical criteria for the detection of pulsed carriers in noise. I, *J. Appl. Phys.* **24,** 371 (1953). M **14,** 1105

Middleton, D., Further remarks on the nature of the statistical observer, *J. Appl. Phys.* **25,** 127 (1954). M **15,** 728

Middleton, D., Information loss attending the decision operation in detection, *J. Appl. Phys.* **25,** 127 (1954). M **15,** 728

♦Middleton, D., Discussion of "Statistical criteria for the detection of pulsed carriers in noise I, II", *J. Appl. Phys.* **25,** 128 (1954). M **15,** 887

♦Middleton, D., (See W. B. Davenport), *J. Appl. Phys.* **23,** 377 (1952).

Mineur, H., Nouvelle méthode de lissage et méthode pour déterminer avec précision la période d'un phénomène, *Ann. Astrophysique* **6,** 136 (1943). M **8,** 55

Mittmann, O. M. J., Varianz-Untersuchungen bei stationären Variablen, *Arch. Meteorol. Geophys. Bioklimatol.* **9,** 519 (1957). M **18,** 959

Mittmann, O. M. J., Betrachtungen zur Analyse empirischer Funktionen, *Z. Angew, Math. Mech.* **34,** 37 (1954). M **15,** 886

♦Moore, G. H., Time series significance tests based on signs of differences, *J. Amer. Statist. Assoc.* **38,** 153 (1943). M **4,** 281

Moran, P. A. P., Some theorems on time series, I, *Biometrika* **34,** 281 (1947). M **9,** 361

Moran, P. A. P., Some theorems on time series. II. The significance of the serial correlation coefficient, *Biometrika* **35,** 255 (1948). M **10,** 385

Moran, P. A. P., The oscillatory behaviour of moving averages, *Proc. Cambridge Philos. Soc.* **46,** 272 (1950). M **11,** 672

Morrison, J. T., Note on the correlation of time-series, *Philos. Mag.* VII **18,** 545 (1934). Z **10,** 72

Mosetti, F., Su di un metodo di analisi delle periodicità, *Ann. Geofis.* **8,** 331 (1955). M **17,** 641

Murteira, B., Note on the variate differences of autoregressive series, *Biometrika* **38,** 479 (1951). M **13,** 855

♦Murty, V. K., (See K. Nagabhushanam), *Calcutta Statist. Assoc. Bull.* **5,** 135 (1954).

Nagabhushanam, K., Some aspects of stationary time series, *Sankhyā* **12,** 109 (1952). M **14,** 998

♦Nagabhushanam, K., On equally correlated stationary processes, *Calcutta Statist. Assoc. Bull.* **5,** 135 (1954). M **16,** 273

Neyman, J., On a new class of "contagious" distributions, applicable in entomology and bacteriology *Ann. Math. Statist.* **10,** 35 (1939). Z **20,** 382

Niessen, A. M., On the summation of certain progressions useful in time series analysis, *J. Amer. Statist. Assoc.* **40,** 98 (1945). M **6,** 234

Noether, G. E., Asymptotic properties of the Wald-Wolfowitz test of randomness, *Ann. Math. Statist.* **21,** 231 (1950). M **11,** 732

Odier, M., Les moments infinitésimaux et le problème du risque dans les caisses de pensions autonomes, *Thesis. University of Zürich* (1945). M **12,** 271

Ono, K., Eine Ausgleichungsmethode der statistischen Reihen, *Jap. J. Math.* **17,** 117 (1940). M **3,** 156

Ono, K., Über eine Art der Ausgleichung der statistischen Reihen, *Jap. J. Math.* **17,** 513 (1941). M **3,** 175

Ostrowski, A., La recherche des périodicités cachées, *Colloq. Internat. Centre Nat. Rech. Sci.* **15,** 93, *Paris* (1949). M **11,** 405

Parzen, E., On consistent estimates of the spectral density of a stationary time series, *Proc. Nat. Acad. Sci. U.S.A.* **42,** 154 (1956). M. **17,** 872

Patankar, V. N., The goodness of fit of frequency distributions obtained from stochastic processes, *Biometrika* **41,** 450 (1954). M **16,** 731

♦Peterson, W. W., (See D. Middleton), *J. Appl. Phys.* **25,** 128 (1945).

Pierce, J. A., On the summation of progressions useful in time series analysis, *J. Amer. Statist. Assoc.* **39,** 387 (1944). M **6,** 238

♦Pope, J. A., (See F. H. C. Marriott), *Biometrika* **41,** 390 (1954).

Preston, G. W., On the theory of prediction of non-stationary stochastic processes, *J. Appl. Phys.* **24,** 230 (1953) M **14,** 667

Quenouille, M. H., A large-sample test for the goodness of fit of autoregressive schemes, *J. Roy. Statist. Soc.* **110,** 123 (1947). M **9,** 603

Quenouille, M. H., Approximate tests of correlation in time-series, *Proc. Cambridge Philos. Soc.* **45,** 483 (1949). M **10,** 724

Quenouille, M. H., Approximate tests of correlation in time-series, *J. Roy. Statist. Soc.* (*Ser. B*) **11,** 68 (1949). M **11,** 262

Quenouille, M. H., On a method of trend elimination, *Biometrika* **36,** 75 (1949). M **11,** 674

Quenouille, M. H., Trend elimination in time-series, *Bull. Inst. Internat. Statist.* **32,** 290 (1950). M **13,** 261

Quenouille, M. H., Modifications to the variate-difference method, *Biometrika* **40,** 383 (1953). M **15,** 454

♦Ragazzini, J. R., (See L. A. Zadeh), *J. Appl. Phys.* **21,** 645 (1950).

♦Rajalakshman, D. V., (See M. S. Bartlett), *J. Roy. Statist. Soc.* (*Ser. B*) **15,** 107 (1953).

Rao, C. R., The applicability of large sample tests for moving average and autoregressive schemes to series of short length—an experimental study. Part 3: The discriminant function approach in the classification of time series, *Sankhyā* **11,** 257 (1951). M **14,** 391

♦Rao, S. R., The applicability of large sample tests for moving average and autoregressive schemes to series of short length—an experimental study. Part 2: Autoregressive series, *Sankhyā* **11,** 239 (1951). M **14,** 391

Reiersol, O., Confluence analysis by means of lag moments and other methods of confluence analysis, *Econometrica* **9,** 1 (1941). M **2,** 237

Rice, S. O., Mathematical analysis of random noise, *Bell System Tech. J.* **23,** 282 (1944). M **6,** 233

Rice, S. O., Mathematical analysis of random noise, *Bell System Tech. J.* **24,** 46 (1945). M **6,** 89

♦Richards, O. W., The course of population growth and the size of seeding, *Growth* **1,** 217 (1937). Z **20,** 151

Robin, L., Fonction de corrélation propre et spectre de la densité de puissance du bruit thermique écrêté. Filtrage de signaux périodiques, simples, dans ce bruit, *Ann. Télécommun.* **7,** 375 (1952). M **15,** 974

♦Robinson, E. A., (See G. P. Wadsworth), *Geophysics* **18,** 539 (1953).

Rosenblatt, M., On the estimation of regression coefficients of a vector-valued time series with a stationary residual, *Ann. Math. Statist.* **27,** 99 (1956). M **17,** 871

♦Rosenblatt, M., (See U. Grenander), *Trans. Amer. Math. Soc.* **76,** 112 (1945).

♦Rosenblatt, M., (See U. Grenander), *Proc. Nat. Acad. Sci. U.S.A.* **40,** 812 (1954).

♦Rosenblatt, M., (See U. Grenander), *Proc. Nat. Acad. Sci. U.S.A.* **38,** 519 (1952).

♦Rosenblatt, M., (See U. Grenander), John Wiley & Sons, N.Y. (1957).

♦Rosenblatt, M., (See U. Grenander), *Proc. Symp. Math. Statist. Prob.*, **1,** 77, U. Calif. Press, Berkeley (1956).

♦Rosenblatt, M., (See U. Grenander), *Ann. Math. Statist.* **24,** 537 (1953).

♦Rosenblatt, M., (See U. Grenander), *Skand. Aktuarietidskr.* **36,** 182 (1953)

Rudra, A., A method of discrimination in time series analysis. I., *Sankhyā* **15,** 9 (1955). M **17,** 170

Rudra, A., A method of discrimination in time series analysis. II, *Sankhyā* **17,** 51 (1956). M **18,** 683

Rudra, A., A critical survey of some test methods in time series analysis, *Calcutta Statist. Assoc. Bull.* **5,** 165 (1954). M **16,** 731

Salvemini, T., Contributo allo studio della mutabilitá delle serie cicliche, *Metron* **15,** 29 (1949). M **12,** 118

Sargan, J. D., An approximate treatment of the properties of the correlogram and periodogram, *J. Roy. Statist. Soc.* (*Ser. B*) **15,** 140 (1953). M **15,** 811

Sastry, A. S. R., Some moments of moment statistics and their use in tests of significance in autocorrelated series, *Sankhyā* **11,** 297 (1951). M **14,** 66

Sastry, A. S. R., Bias in estimation of serial correlation coefficients, *Sankhyā* **11,** 281 (1951). M **14,** 66

Savur, S. R., A test of significance in approximate periodogram analysis, *Sankhyā* **6,** 77 (1942). M **4,** 108

Schumann, T. E. W., A mechanical appliance for the smoothing of time series, *Philos. Mag.* **30,** 39 (1940). M **2,** 62

Schumann, T. E. W., On Yule's Method of investigating periodicities of disturbed series. The motion of a pendulum in a turbulent fluid, *Philos. Mag.* **33,** 138 (1942). M **4,** 27

♦Seguchi, T., Note on the statistical inferences of certain continuous stochastic processes, *Mem. Fac. Sci. Kyūsyū Univ.* (*A*) **8,** 187 (1954). M **16,** 385

Smith, J. L. S., The specification of disturbed periodic time series of the type of Wolfer's annual sunspot numbers, *J. Roy. Statist. Soc.* **107,** 231 (1944). M **7,** 133

♦Smith, W. L. (See D. R. Cox), *Biometrika* **40,** 1 (1953).

Solodovnikov, V. V., Analysis and synthesis of tracking and regulating systems subject to stationary random disturbances, *Izvestiya Akad. Nauk. SSSR* **1950,** 1648 (1950). M **12,** 430.

♦Som, R. K. (See S. R. Rao), *Sankhyā* **11,** 239 (1951)

Spencer-Smith, J. L., The oscillatory properties of the moving average, *Suppl. J. Roy. Statist. Soc.* **9,** 104 (1947). M **9,** 363

Spetner, L. M., Errors in power spectra due to finite sample, *J. Appl. Phys.* **25,** 653 (1954). M **15,** 887

Sreedharan Pillai, K. C., Trend analyser, *Proc. Indian Acad. Sci.* (*Sec. A*) **17,** 187 (1943). M **5,** 51

Starkey, D. M., The distribution of the multiple correlation coefficient in periodogram analysis, *Ann. Math. Statist.* **10,** 327 (1939). M **1,** 152

Stone, J. R. N., Prediction from autoregressive schemes and linear stochastic difference systems, *Econometrica* **17** *Suppl.*, 29 (1949). M **12,** 512

Storch, J. M., The use of moving averages for the analysis of an economic graph, *Statistica. Rijswijk* **4,** 37 (1950). M **12,** 194

Strecker, H., Die Quotientenmethode, eine Variante der "Variate Difference" Methode, *Mitteilungsblatt Math. Statist.* **1,** 115 (1949). M **11,** 449

♦Stuart, A. (See F. G. Foster), *J. Roy. Statist. Soc.* (*Ser. B*) **16,** 1 (1954).

Stumpers, F. L., *A bibliography of information theory. Communication theory—cybernetics*, (Res. Lab. Electronics, Mass. Inst. Tech. Cambridge, 1953). M **15,** 638

Suryanarayana, O., Estimation and tests of significance of the components of a time-series, *Sankhyā* **15,** 303 (1955). [M **17,** 280

Tager, P. G., The frequency spectrum in phase-pulse modulation, *Avtomatika i Telemehanika* **8,** 117 (1947). M **11,** 732

Teatini, U., Interpolazione di serie cicliche, *Statistica. Milano* **8,** 125 (1948). M **11,** 530

♦Teichroew, D. (See F. G. Foster), *J. Roy. Statist. Soc.* (*Ser. B*) **17,** 115 (1955).

Theil, H., On "upcrosses" and "downcrosses" in time series, *Math Centrum Amsterdam Rapport ZW* **1950** 010 (1950). M **12,** 347

Tintner, G., The Variate Difference Method, *Mono. 5, Cowles Comm.* Principia Press, Bloomington, Ind. (1940). M **1,** 250

Tintner, G., An application of the variate difference method to multiple regression, *Econometrica* **12,** 97 (1944). M **6,** 93

Tintner, G., A note on rank, multicollinearity and multiple-regression, *Ann. Math. Statist.* **16,** 304 (1945). M **7,** 132

Turner, C. H. M., On the concept of an instantaneous power spectrum, and its relationship to the autocorrelation function. *J. Appl. Phys.* **25,** 1347 (1954). M **16,** 499

van Woerkom, A. J. J., On cumulative sums of random numbers. *Astr. J.* **58,** 10 (1953). M **15,** 45

♦Vialar, J., (See M. Kivéliovitch), *J. Sci. Météorol.* **6,** 151 (1954).

♦Vialar, J., (See M. Kivéliovitch), *J. Sci. Météorol.* **6,** 73 (1954).

♦Vialar, J., (See M. Kivéliovitch), *J. Sci. Météorol.* **5,** 21, 75, 89, 129 (1953).

Vickery, C. W., Cyclically invariant graduation, *Econometrica* **12,** 19 (1944). M **5,** 246

von Schelling, H., Kennzeichen für eine rein zufällige Folge der Werte in einer zeitlich geordneten Beobachtungsreihe, *Astr. Nachr.* **269,** 155 (1939). M **1,** 249

♦Wadsworth, G. P., Detection of reflections on seismic records by linear operators, *Geophysics* **18,** 539 (1953). M **15,** 728

Wald, A., Long cycles as a result of repeated integration, *Amer. Math. Monthly* **46,** 136 (1939). Z **21,** 45

♦Wald, A., Optimum character of the sequential probability ratio test, *Ann. Math. Statist.* **19,** 326 (1948). M **10,** 201

♦Wald, A., (See H. B. Mann), *Econometrica* **11,** 173 (1943).

Walker, A. M., Note on a generalization of the large sample goodness of fit test for linear autoregressive schemes, *J. Roy. Statist. Soc.* (*Ser. B*) **12,** 102 (1950). M **12,** 512

Walker, A. M., A goodness of fit test for spectral distribution functions of stationary time series with normal residuals, *Biometrika* **43,** 257 (1956). M **18,** 522

Walker, A. M., The asymptotic distribution of serial correlation coefficients for autoregressive processes with dependent residuals, *Proc. Cambridge Philos. Soc.* **50,** 60 (1954). M **15,** 331

♦Wallis, W. A., (See G. H. Moore), *J. Amer. Statist. Assoc.* **38,** 153 (1943).

Wallis, W. A., A significance test for time series, *Tech. Paper No.* 1 *Nat. Bur. Econ. Research*, N.Y. (1941). M **3,** 176

♦Wallis, W. A., (See C. Eisenhart), McGraw-Hill Book Co., Inc., New York (1947).

Watson, G. S., On the joint distribution of the circular serial correlation coefficients, *Biometrika* **43,** 161 (1956). M **18,** 79

Wescott, J. H., The minimum-moment-of-error-squared criterion: a new performance criterion for servo mechanisms, *Proc. Inst. Elec. Engrs. Part* II **101,** 471 (1954). M **16,** 386

Westenberg, J., Mathematics of pollen diagrams, *Nederl. Akad. Wentensch, Proc.* **50,** 509 and 640 (1947). M **9,** 151

Whittle, P., Some results in time series analysis, *Skand. Aktuarietidskr.* **35,** 48 (1952). M **14,** 191

Whittle, P., The simultaneous estimation of a time series harmonic components and covariance structure, *Trabajos Estadística* **3,** 43 (1952). M **14,** 488

Whittle, P., On stationary processes in the plane, *Biometrika* **41,** 434 (1954). M **16,** 731

Whittle, P., Hypothesis Testing in Time Series Analysis, *Thesis, Uppsala University* (1951). M **12,** 726

Whittle, P., Tests of fit in time series, *Biometrika* **39,** 309 (1952). M **14,** 666

Whittle, P., Certain nonlinear models of population and epidemic theory, *Skand. Aktuarietidskr.* **35,** 211 (1952). M **14,** 890

Whittle, P., The analysis of multiple stationary time series, *J. Roy. Statist. Soc.* (*Ser. B*) **15,** 125 (1953). M **15,** 143

Whittle, P., Estimation and information in stationary time series, *Ark. Mat.* **2,** 423 (1953). M **15,** 728

Wiener, N., *Cybernetics, or control and communication in the animal and the machine*, (John Wiley & Sons, New York, 1948). M **9,** 598

Wiener, N., *Extrapolation, Interpolation, and Smoothing of Stationary Time Series. With Engineering Applications* (John Wiley & Sons, N.Y. 1949). M **11,** 118

Wilson, E. B., The standard deviation of sampling for life expectancy, *J. Amer. Statist. Assoc.* **33,** 705 (1938). Z **20,** 47

Wise, J., Stationarity conditions for stochastic processes of the autoregressive and moving-average type, *Biometrika* **43,** 215 (1956). M **17,** 1221

Wise, J., The autocorrelation function and the spectral density function, *Biometrika* **42,** 151 (1955). M **16,** 1040

Wold, H., *A study in the analysis of stationary time series. 2d ed. With an appendix by Peter Whittle*, (Almqvist and Wiksell, Stockholm, 1954). M **15,** 811

Wold, H., Sur les processus stationnaires ponctuels, *Colloq. Int. Centre Nat. Rech. Sci.* **13,** 75, *Paris* (1949). M **11,** 258

Wold, H., A large-sample test for moving averages, *J. Roy. Statist. Soc.* (*Ser. B*) **11,** 297 (1949). M **11,** 674

Wold, H., On least square regression with autocorrelated variables and residuals, *Bull. Inst. Internat. Statist.* **32,** 277 (1950). M **13,** 261

Wold, H. O. A., Series cronologicas estacionarias, (Stationary Time Series), *Monografias No.* 28, *Cons. Investig. Cientificas, Madrid* (1951). M **13,** 368

♦Wold, H., (See R. Bentzel), *Skand. Aktuarietidskr.* **29,** 95 (1946).

♦Wolfowitz, J., (See A. Wald), *Ann. Math. Statist.* **19,** 326 (1948).

♦Wolfowitz, J., (See A. Dvoretzky), *Ann. Math Statist.* **24,** 254 (1953).

♦Wolfowitz, J., (See H. Levene), *Ann. Math Statist.* **15,** 58 (1944).

Woodward, P. M., Time and frequency uncertainty in waveform analysis, *Philos. Mag.* **42,** 883 (1951). M **13,** 167

Zadeh, L. A., An extension of Wiener's theory of prediction, *J. Appl. Phys.* **21,** 645 (1950). M **12,** 347

♦Zaremba, S. K., (See Z. A. Lomnicki), *J. Roy. Statist. Soc.* (*Ser. B*) **17,** 243 (1955).

Zasuhin, V., On the theory of multidimensional stationary random processes, *C. R. Doklady Acad. Sci URSS* **33,** 435 (1941). M **5,** 102

WASHINGTON, D.C. (Paper 64B1–23)

Publications of the National Bureau of Standards*

(Including Papers in Outside Journals)

Selected Abstracts

Use of Chebychev polynomials in thin film computations, K. D. Mielenz, *J. Research NBS* **63A,** *No. 3, 297 (1959).*

From Herpin's expression for the mth power of a multilayer matrix, very simple closed formulas are derived for the matrices and optical constants of any multilayer with a periodic structure.

According to Epstein's theorem, any symmetrical multilayer is equivalent to a fictitious monolayer. A simple expression for the equivalent index and thickness of this monolayer is deduced for the case of a periodic and symmetrical sequence of equally thick films.

As compared to any other method of numerical computation, the suggested formulation provides a considerable saving of time and work. In a numerical example, this saving amounts to about 80 percent.

Theory of formation of polymer crystals with folded chains in dilute solution, J. I. Lauritzen, Jr., and J. D. Hoffman, *J. Research NBS* **64A,** *No. 1, 73 (1960).*

A detailed interpretation of the kinetics of homogeneous nucleation and growth of a polymer from dilute solutions is given. The probability of forming both nuclei with folded chains, and the conventional bundle-like nuclei, from dilute solution is carefully analyzed. It is predicted that at sufficiently high dilution, critical nuclei of length $\mathbf{l}^*$ will be formed from single polymer molecules by sharp folding of the chain backbone. After growth, the resulting crystal is flat and platelike, the loops formed by the chain folds being on the upper and lower surfaces. Kinetic factors determine that the distance between the flat surfaces in the grown crystal will vary over only a narrow range of values about a characteristic step height, $\mathbf{l}^*=\sigma_e/\Delta\mathbf{f}$. Here σ_e is the free energy required to form a unit area of the flat surface, and possesses a rather large value because the work involved in forming folds is included. The quantity $\Delta\mathbf{f}$ is the free energy difference per unit volume of crystal between the crystalline and solution states, and is approximately proportional to the degree of supercooling, $\Delta\mathbf{T}$. These crystals are metastable. The logarithm of the nucleation rate is approximately proportional to $1/(\Delta\mathbf{T})^2$ near the melting point. The exponent n in the free growth rate law is predicted under various assumptions. To the extent that comparison is possible, the predictions given agree with the experimental results obtained by Keller and O'Connor and others on single crystals of unbranched polyethylene grown from dilute solution.

Effect of antenna size on gain, bandwidth, and efficiency, R. G. Harrington, *J. Research NBS* **64D,** *No. 1, 1 (1960).*

A theoretical analysis is made of the effect of antenna size on parameters such as gain, bandwidth, and efficiency. Both near-zone and far-zone directive gains are considered. It is found that the maximum gain obtainable from a broadband antenna is approximately equal to that of the uniformly illuminated aperture. If higher gain is desired, the antenna must necessarily be a narrow-band device. In fact, the input impedance becomes frequency sensitive so rapidly that, for large antennas, no significant increase in gain over that of the uniformly illuminated aperture is possible. Also, if the antenna is lossy, the efficiency falls rapidly as the gain is increased over that of the uniformly illuminated aperture.

Surface-wave resonance effect in a reactive cylindrical structure excited by an axial line source, A. L. Cullen, *J. Research NBS* **64D,** *No. 1, 13 (1960).*

It is shown that a purely reactive cylinder excited by a neighboring line source can, under suitable conditions, give rise to a radiation pattern closely approximating the function $\cos n\theta$.

In a numerical example, a cylinder of three transverse electromagnetic wavelengths circumference has a surface reactance chosen to emphasize the term $\cos 6\theta$ in the Fourier series of the resultant radiation pattern. It is shown that only 1.1 percent of the total power delivered to the line source is radiated in unwanted modes.

It is also shown that the position of the line source does not affect this result to first order provided that $k(b-a)<<1$, where $b-a$ is the distance of the line source from the cylindrical surface.

An exact earth-flattening procedure in propagation around a sphere, B. Y. -C. Koo and M. Katzin, *J. Research NBS* **64D,** *No. 1, 61 (1960).*

By a refinement of the procedure used in the usual earth-flattening approximation, the problem of propagation around a spherical earth is reduced to an exact equation of the same form. Thereby the earth-flattening procedure becomes applicable to arbitrarily large heights and distances. It is also found that existing solutions of the approximate equations can be reevaluated to yield the exact solutions for slightly different refractive index distributions.

Some applications of statistical sampling methods to outgoing letter mail characteristics, N. C. Severo, A. E. Newman, S. M. Young, and M. Zelen, *NBS Tech. Note 16 (PB151375) $2.75.*

This paper presents applications of statistical sampling procedures especially devised to procure information about the characteristics of outgoing letter mail. The results of four separate studies carried out in the Washington, D.C., San Francisco, and Los Angeles post offices are herein summarized. The techniques used in the various studies were developed so that the required information would be of predetermined reliability and could be gathered without the use of a large staff and without interrupting the flow of mail. The four studies presented concern: (1) Letter size and color characteristics, (2) ratio of hand canceled mail to machine canceled mail, (3) top and bottom clearance space of an addressed envelope, (4) proportions of long and short letters.

Terrestrial propagation of very-low-frequency radio waves, J. R. Wait, *J. Research NBS* **64D,** *No. 2, 153 (1960).*

A self-contained treatment of the wave-guide mode theory of the propagation of very-low-frequency radio waves is presented. The model of a flat earth with a sharply bounded and homogeneous ionosphere is treated for both vertical and horizontal dipole excitation. The properties of the modes are discussed in considerable detail.

The influence of earth curvature is also considered by reformulating the problem using spherical wave functions of complex order. The modes in such a curved guide are investigated and despite the initial complexity of the general solution, many interesting and limiting cases may be treated in simple fashion to yield useful and convenient formulas for calculation.

Other factors considered are the influence of the earth's magnetic field, antipodal effects, resonator type oscillations, and the influence of stratification at the lower edge of the ionosphere.

Inclusion theorems for congruence subgroups, M. Newman and I. Reiner, *Trans. Am. Math. Soc.* **91,** *No. 3, 369 (1959).*

Let G_t be the group consisting of all $t \times t$ integral matrices of determinant 1. For a fixed partition $t=r+s$ of t into positive integers r, s, and for a fixed positive integer n, define the subgroup

$$G_{r,s}(n) = \left\{ \begin{pmatrix} A^{(r)} & B \\ C & D^{(s)} \end{pmatrix} \epsilon G_t : \ C \equiv 0 \pmod{n} \right\}.$$

Suppose that m is a positive integer and H a group such that

$$G_{r,s}(mn) \subset H \subset G_{r,s}(n).$$

Then

$$H = G_{r,s}(dn) \text{ where } d|m.$$

Further results of this type are obtained.

New approach in the theory of satellite orbits, J. P. Vinti, *Phys. Rev. Letters* **3,** *No. 1, 8 (1959).*

A gravitational potential has been found, expressed in oblate spheroidal coordinates, which results in separability of the Hamilton-Jacobi equation for the motion of an earth satellite and which is much closer to the empirically accepted one than any heretofore used as the starting point of a calculation. It thus becomes possible to do orbit theory for unretarded satellites very accurately without the use of perturbation theory.

Calculated patterns of slotted elliptic-cylinder antennae, J. R. Wait and W. E. Mientka, *Appl. Sci. Research* **B7,** *449 (1959).*

The model assumed is a perfectly conducting elliptic cylinder of infinite length which has a narrow axial slot of finite length. Patterns are presented for various ratios of the major to minor axis. It is indicated that for reasonably large elliptic cylinders, the patterns are dependent mainly on the surface curvature in the neighborhood of the slot.

On prediction of system behavior, J. R. Rosenblatt, *Proc. of the N.Y. Univ.–Ind. Conf. on Reliability Theory (New York, N.Y.) p. 39 (1958).*

It is customary in theoretical discussions of prediction of system performance to assume that the relation between variables by which performance is assessed and variables describing the parts of the system can be specified by a function. The purpose of this paper is to consider some of the problems which arise in choosing the form of such a function and in specifying the relevant variables and the nature of appropriate experiments, measurements, and data.
In the context of each of a number of familiar types of mathematical model, the possible consequences of alternative choices of time units, subsystem definitions, and independence assumptions are discussed.

Kinetic equation for a plasma with unsteady correlations, C. M. Tchen, *Phys. Rev.* **114,** *394 (1959).*

As a generalization of the Boltzmann equation, the kinetic equation for a plasma is derived in the form of a generalized Fokker-Planck equation, by considering unsteady correlations, including non-Markovian and nonlinear behavior. Both the binary and ternary correlations are used for many kinds of particles with different temperatures. The coefficients of the kinetic equation depend on the law of interaction for a pair of particles and are influenced by relaxation. The effective potential of friction consists of two parts: the static part corresponds to the Debye potential and is isotropic, the dynamical part is axially symmetrical about the direction of motion, and causes a dynamical friction. The results show that the friction is proportional to velocity for slow particles and inversely proportional to the square of velocity for fast particles. This tendency of the fast particles to overcome repulsion is a property connected with the "run-away" of electrons. A criterion for maximum friction is derived. The triplet interaction, which mainly affects the shielding phenomena, assures the convergence of the coefficients in case of distant interaction. Since the length scales of interaction are well determined in this way, the kinetic equation can be expected to be valid over a longer range than does the Boltzmann equation. The large scale agrees with the Debye radius when the shielding term is linearized, as should be expected. When time relaxation is left aside and linearization is made, the kinetic equation degenerates to the classical Fokker-Planck equation with convergent coefficients.

List of Titles

Journal of Research, Section 63A, No. 3, November-December 1959. 70 cents.

Multiple ionization of rare gases by electron impact. M. Krauss, R. M. Reese, and V. H. Dibeler.
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Low even configurations in the first spectrum of ruthenium (Ru I), part 2. R. E. Trees.
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A preliminary list of levels and g-values for the first spectrum of thorium (Th I). Romuald Zalubas.
OH in the solar spectrum. Charlotte E. Moore and Herbert P. Broida.
Use of Chebychev polynomials in thin film computations. Klaus D. Mielenz. (See above abstracts.)

Journal of Research, Section 64A, No. 1, January-February 1960. 70 cents.

The "1958 He^4 scale of temperatures." F. G. Brickwedde, H. van Dijk, M. Durieux, J. R. Clement, and J. K. Logan.
Energy levels and spectrum of neutral helium (^4He I). William C. Martin.
Vibration-rotation structure in absorption bands for the calibration of spectrometers from 2 to 16 microns. Earle K. Plyler, Alfred Danti, L. R. Blaine, and E. D. Tidwell.
Combustion calorimetry with fluorine: constant pressure flame calorimetry. G. T. Armstrong and R. S. Jessup.
Pits in metals caused by collision with liquid drops and rigid steel spheres. Olive G. Engel.
Theory of formation of polymer crystals with folded chains in dilute solution. John I. Lauritzen, Jr., and John D. Hoffman. (See above abstracts.)
Studies of beryllium chromite and other beryllia compounds with R_2O_3 oxides. C. E. Weir and A. Van Valkenburg.
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Impedance of a corner-reflector antenna as a function of the diameter and length of the driven element. A. C. Wilson.
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Terrestrial propagation of very-low-frequency radio waves, a theoretical investigation. James R. Wait.
Aurora of October 22/23, 1958, at Rapid City, South Dakota. F. E. Roach and E. Marovich.

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The changing character of chemical research in government, E. Wichers, The Chemist XXXVI, No. 7, 260 (1959).
Study of the setting of plaster, K. D. Jorgensen and A. S. Posner, J. Dental Research **38,** No. 3, 491 (1959).

Radiation attenuation data, H. O. Wyckoff, Radiation Hygiene Handbook, edited by H. Blatz, Section 8, p. 1 (McGraw-Hill Book Co., New York, N.Y., 1959).

Corrosion of type 310 stainless steel by synthetic fuel oil ash, H. L. Logan, Corrosion **15,** 443t (1959).

Physical research, Part 2, G. C. Paffenbarger and W. Souder, J. Am. Dental Assoc. **58,** 98 (1959).

Effects of gamma radiation on collagen, J. Cassel, J. Am. Leather Chemists' Assoc. LIV, No. 8, 432 (1959).

Standards for neutron flux measurement and neutron dosimetry, R. S. Caswell, E. R. Mosburg, Jr., and J. Chin, 2d United Nations International Conf. on the Peaceful Uses of Atomic Energy, Vol. 21, Health and Safety: Dosimetry and Standards, P/752 USA, p. 92 (1959).

Absorption of radiation by a cylindrical sample of a strong absorber, P. H. Fang and I. A. Stegun, Letter to Editor, J. Chem. Phys. **31,** 267 (1959).

PILOT—A new multiple computer system, A. L. Leiner, W. A. Notz, J. L. Smith, and A. Weinberger, J. Assoc. Computing Mach. **6,** No. 3, 313 (1959).

Properties of rutile (titanium dioxide), F. A. Grant, Rev. Modern Phys. **31,** 646 (1959).

Analysis of vibrational relaxation data in shock wave experiments, K. E. Shuler, J. Chem. Phys. **30,** No. 6, 1631 (1959).

The interlaboratory evaluation of testing methods, J. Mandel and T. W. Lashof, ASTM Bull. No. 239, 53 (1959).

Branched-chain higher sugars. I. A 9-aldo-4-*C*-formyl-nonose derivative, R. Schaffer and H. S. Isbell, J. Am. Chem. Soc. **81,** 2178 (1959).

The structure of electrolytic solutions, W. J. Hamer (John Wiley & Sons, Inc., New York, N.Y., 1959).

The use of a vacuum microbalance in studies of electron tube materials, G. F. Rouse, Proc. 4th Natl. Conf. on Tube Techniques, September 10–12, 1958, sponsored by the Advisory Group on Electron Tubes, p. 262 (New York University Press, New York, N.Y., 1959).

Widths of cracks in concrete at the surface of reinforcing steel evaluated by means of tensile bond specimens, D. Watstein and R. G. Mathey, J. Am. Concrete Inst. **31,** No. 1, 47 (1959).

Exposure standards and radiation protection regulations, L. S. Taylor, Radiation Hygiene Handbook, edited by H. Blatz, Section 3, p. 2 (McGraw-Hill Book Co., New York, N.Y., 1959).

Geometrical anisotropy of magnetic materials in wave guides and cavities, A. L. Steinert, J. Appl. Phys. **30,** 1109 (1959).

Radioactivity standardization in the United States, W. B. Mann and H. H. Seliger, 2d U.N. Intern. Conf. on the Peaceful Uses of Atomic Energy, Vol. 21, Health and Safety: Dosimetery and Standards, P/750 USA, p. 90 (1959).

Magnetic interaction of H_3, V. Griffing, J. L. Jackson, and B. J. Ransil, J. Chem. Phys. **30,** No. 4, 1066 (1959).

Program of the International Commission on radiological units and measurements, L. S. Taylor, L. H. Gray, and H. O. Wyckoff, 2d U.N. Intern. Conf. on the Peaceful Uses of Atomic Energy, Vol. 21, Health and Safety: Dosimetry and Standards, P/2243 WHO, p. 81 (1959).

The nature, cause and effect of the porosity in electro-deposits, IV. Influence of gas bubbles on the formation of pores, F. Ogburn and D. W. Ernst, Plating **46,** 957 (1959).

Low even configurations in the first spectrum of ruthenium (Ru I), R. E. Trees, J. Opt. Soc. Am. **49,** 838 (1959).

End plate modification of X-band TE_{011} cavity resonators, M. C. Thompson, Jr., F. E. Freethey, and D. M. Waters, Letter to editor, IRE Trans. on Microwave Theory Tech. **MTT-7,** 388 (July 1959).

Introductory remarks, A. V. Astin, Proc. 5th Tech. Session on Bone Char 1957, p. 1 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Analysis of liquid sugars, E. J. McDonald, Proc. 5th Tech. Session on Bone Char 1957, p. 77 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Development of a new test for the abrasion hardness of bone char, F. G. Carpenter, Proc. 5th Tech. Session on Bone Char 1957, p. 99 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Effects of controlled decarbonization on the performance of service synthad, F. W. Schwer, W. V. Loebenstein, and E. P. Barret, Proc. 5th Tech. Session on Bone Char 1957, p. 145 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Some mechanisms of color and ash removal by bone char, A. Gee, Proc. 5th Tech. Session on Bone Char 1957, p. 163 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Survey of variations in use for conducting laboratory-scale column filteration tests, V. R. Deitz, Proc. 5th Tech. Session on Bone Char 1957, p. 237 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Comparison of column decolorization experiments with theory, W. V. Loebenstein, Proc. 5th Tech Session on Bone Char 1957, p. 253 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Sugar retention by char, F. G. Carpenter, Proc. 5th Tech. Session on Bone Char 1957, p. 279 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Removal of organic anions by bone char, V. R. Ditz and H. M. Rootare, Proc. 5th Tech. Session on Bone Char 1957, p. 297 (Bone Char Research Project, Inc., Charlestown, Mass., 1959).

Recent advances in cryogenic engineering, R. B. Jacobs, Am. Rocket Soc. ARS J. **29,** 245 (1959).

Transistorized velocimeter for measuring speed in the sea, C. E. Tschiegg, J. Acoust. Soc. Am. **31,** No. 7, 1038 (1959).

Magnetic study of the frozen products from the nitrogen microwave discharge, B. J. Fontana, J. Chem. Phys. **31,** No. 1, 148 (1959).

The thermal E. M. F. of several thermometric alloys, R. L. Powell and M. D. Bunch, Suppl. Bull. Inst. Intern. du Froid (Delft, Holland) Comm. **1,** 129 (1958).

Electrical discharge induced luminescence of solids at low temperatures, L. J. Schoen and R. E. Rebbert, J. Mol. Spect. **3,** No. 4, 417 (1959).

The nature, cause and effect of the porosity in electrodeposits. V. An evaluation of the sensitivity of the ferroxyl test, F. Ogburn, D. W. Ernst, and W. H. Roberts, Plating **46,** 1052 (1959).

Spectrum of thin target bremsstrahlung bounded by a forward circular cone, J. H. Hubbell, J. Appl. Phys. **30,** No. 7, 981 (1959).

Negative atomic ions, H. R. Johnson and F. Rohrlich, J. Chem. Phys. **30,** 1068 (1959).

Factors affecting modulation techniques for VHF scatter systems, J. W. Koch, IRE Trans. Commun. Syst. **CS–7,** No. 2, 77 (1959).

The very-low-frequency emissions generated in the earth's exosphere, R. M. Gallet, Proc. IRE **47,** No. 2, 211 (1959).

Reaction of hydrogen atoms with solid oxygen at 20° K., R. Klein and M. D. Scheer, J. Chem. Phys. **31,** No. 1, 278 (1959).

Application of the Williams-Landel-Ferry equation to silicate glasses, A. B. Bestul, Glastech. Ber. (Frankfurt, Germany) **32 K,** No. VI, 59 (1959).

Glow discharge spectra of copper and indium above aqueous solutions, D. E. Couch and A. Brenner, Tech. Notes, J. Electrochem. Soc. **106,** 628 (1959).

Free radical chemistry, J. W. Moyer and A. M. Bass, Chem. and Eng. News, p. 51 (Aug. 24, 1959).

Gaseous heat conduction at low pressures and temperatures, R. J. Corrucini, Vacuum **7, 8,** 19 (Pergamon Press Ltd., London, England, 1959).

Discussion of the papers of Messrs. Satterthwaite and Budne, W. J. Youden, Technometrics **1,** No. 2, 157 (1959).

Production of embossing plates from texture patterns by electroforming methods, J. P. Young and V. A. Lamb, Plating **46,** 1033 (1959).

Method of evaluating the clinical effect of warping a denture: Report of a case, J. B. Woelfel and G. C. Paffenbarger, J. Am. Dental Assoc. **59,** 250 (1959).

The sound transmission loss of some building construction, R. V. Waterhouse, R. D. Berendt, and R. K. Cook, Noise Control **5,** No. 4, 40 (1959).

Cryogenic insulation, R. H. Kropschot, Am. Soc. Heating, Refrig. and Air-Conditioning Engrs. **1,** No. 9, 48 (1959).

End plate modification of X-band TEO11 cavity resonators, M. S. Thompson, F. E. Freethey, and D. M. Waters, IRE Trans. on Microw. Theory Tech. **MTT–7,** 388 (1959).

Evolution of amplified waves leading to transition in a boundary layer with zero pressure gradient, P. S. Klebanoff and K. D. Tidstrom, NASA Tech. Note D–195, 1 (1959).

Microwave reflectometer, G. F. Engen and R. W. Beatty, IRE Trans. on Microw. Theory Tech. **MTT–7,** 351 (1959).

Multiple ionization of sodium vapor by electron impact, V. H. Dibeler and R. M. Reese, J. Chem. Phys. **31,** No. 1, 282 (1959).

Penetration and diffusion of X-rays, U. Fano, L. V. Spencer, and M. J. Berger, Encyclopedia of Phys. **38,** No. 2, 660 (1959).

Problems of the experimenter, W. J. Youden, Natl. Conv. Trans. Am. Soc. Quality Control, p. 41 (1959).

Vibrational intensity distributions in the nitrogen afterglow, U. H. Kurzweg and H. P. Broida, J. Mol. Spec. **3,** No. 4, 388 (1959).

Some characteristics of VLF propagation using atmospheric waveforms, W. L. Taylor and L. J. Lange, Recent Advances in Atmos. Elec. (Proc. 2d Conf. on Atmos. Elec.), Portsmouth, N.H. May 20–23, 1958, p. 609 (Pergamon Press, Inc., New York, N.Y., 1958).

A new aid for the rapid determination of absorption corrections by Albrecht's method, D. K. Smith, Acta Cryst. (Copenhagen, Denmark) **12,** pt. 6, 479 (1959).

Measurements made by matching with known standards, W. J. Youden, W. S. Connor, and N. C. Severo, Technometrics **1,** No. 3, 101 (1959).

Influence of crystallographic orientation on the pitting of iron in distilled water, J. Kruger, J. Electrochem. Soc. **106,** No. 8, 736 (1959).

Evacuated powder insulation for low temperatures, M. M. Fulk, Progress in Cryogenics—1, 65 (Heywood and Co., Ltd., London, England, 1959).

A direct-reading viscometer, M. R. Shafer, Instruments and Control Systems **32,** No. 7, 1044 (1959).

Atomic weights, E. Wichers, ch. 6, Treatise on Anal. Chem., pt. 1, Theory and practice, vol. 1, sec. B, Application of chemical principles, p. 161 (The Interscience Encyclopedia, Inc., New York, N.Y., 1959).

Concept and determination of *p*H, R. G. Bates, ch. 10, Treatise on Anal. Chem., pt. 1, Theory and practice, vol. 1, sec. B. Application of chemical principles, p. 361 (The Interscience Encyclopedia, Inc., New York, N.Y., 1959).

Electrode potentials, R. G. Bates, ch. 9, Treatise on Anal. Chem., pt. 1, Theory and practice, vol. 1, sec. B, Application of chemical principles, p. 319 (The Interscience Encyclopedia, Inc., New York, N.Y., 1959).

Principles and methods of sampling, W. W. Walton and J. I. Hoffman, ch. 4, Treatise on Anal. Chem., pt. 1, Theory and practice, vol. 1, sec. A, Analytical chemistry: Its objectives, functions and limitations, p. 67 (The Interscience Encyclopedia, Inc., New York, N.Y., 1959).

System loss in radio-wave propagation, K. A. Norton, Letter to Editor, Proc. IRE **47,** No. 9, 1661 (1959).

The oxide films formed on copper single crystal surfaces in pure water. 1. Nature of the films formed at room temperature, J. Kruger, J. Electrochem. Soc. **106,** No. 10, 847 (1959).

Strain gauge calibration device for extreme temperatures, R. M. McClintock, Rev. Sci. Instr. 30, No. 8, 715 (1959).

Variation of the thermodynamic ideal temperature in the polystyrene-cyclohexane system, D. McIntyre, J. H. O'Mara, and B. C. Konouck, J. Am. Chem. Soc. **81,** 3498 (1959).

Rapid insertion device for coaxial attenuators, A. Y. Rumfelt and R. J. Como, Rev. Sci. Instr. **30,** No. 8, 687 (1959).

Energy dissipation in standing waves in rectangular basins, G. H. Keulegan, J. Fluid Mech. **6,** pt. 1, 35 (1959).

Trapped energetic radicals, J. L. Franklin and H. P. Broida, Ann. Rev. Phys. Chem. **10,** 145 (1959).

Electrode potentials, p. 319, and Concept and determination of *p*H, p. 361, R. G. Bates, pt. 1, vol. 1, Treatise on Anal. Chem., edited by I. M. Kolthoff and P. J. Elving (Interscience Pub. Co., New York, N.Y., 1959).

Total photoelectric cross sections of copper, molybdenum, silver, tantalum, and gold at 662 kev, W. F. Titus, Phys. Rev. **115,** No. 2, 351 (1959).

Diode in feedback loop makes stable transistor bias circuit, G. F. Montgomery, Electronic Design **7,** 54 (1959).

Strength of synthetic single crystal sapphire and ruby as a function of temperature and orientation, J. B. Wachtman and L. H. Maxwell, J. Am. Ceram. Soc. **42,** No. 9, 432 (1959).

PILOT, the NBS multicomputer system, A. L. Leiner, W. A. Nots, J. L. Smith, and A. Weinberger, Proc. Eastern Joint Computer Conf. (Philadelphia, Pa.) p. 164 (1958).

Investigation of creep behavior of structural joints under cyclic loads and temperatures, L. Mordfin, N. Halsey, and G. E. Greene, NASA Tech. Note D. 181, p. 37 (1959).

Dimensional stability of impregnated sole leather, B. H. Fouquet, J. Am. Leather Chemists' Assoc. **LIV,** No. 10, 544 (1959).

Theoretical problems of reliability measurement and prediction, J. R. Rosenblatt, Proc. of the Third Exploratory Conf. on Missile Model Design for Reliability Prediction at White Sands Missile Range (New Mexico) p. 29 (1959).

Accuracy and precision: evaluation and interpretation of analytical data, W. J. Youden, ch. 3, Treatise on Anal. Chem., pt. I. Theory and Practice, vol. 1, sec. A, Analytical chemistry: its objectives, functions and limitations, p. 47 (The Interscience Encyclopedia, Inc., New York, N.Y., 1959).

Dimensional changes in systems of fibrous macromolecules: Polyethylene, L. Mandelkern, D. E. Roberts, A. F. Diorio, and A. S. Posner, J. Am. Chem. Soc. **81,** 4148 (1959).

Microwave reflectometer techniques, G. F. Engen and R. W. Beatty, IRE Trans. on Microw. Theory and Tech. **MTT–7,** No. 3, 346 (1959).

Attenuation of scattered cesium—137 gamma rays, F. S. Frantz, Jr., and H. O. Wyckoff, Radiology **73,** No. 2, 263 (1959).

Observations of the ionosphere over the south geographic pole, R. W. Knecht, J. Geophys. Research **64,** No. 9, 1243 (1959).

Dependence of mechanical strength of brittle polycrystaline specimens on porosity and grain size, F. P. Knudsen, J. Am. Ceram. Soc. **42,** No. 8, 376 (1959).

Theory of dielectric relaxation in molecular crystals, J. D. Hoffman, Archives des Sciences, Soc. Phys. Hist. Nat. Geneve **12,** 36 (1959).

Denture reliners—direct, hard, self-curing resin, G. M. Brauer, E. E. White, Jr., C. L. Burns, and J. B. Woelfel, J. Am. Dental Assoc. **59,** 270 (1959).

Photoproton cross sections of carbon, S. Penner and J. E. Leiss, Phys. Rev. **114,** No. 4, 1101 (1959).

Cryogenic engineering of hydrogen bubble chambers, B. W. Birmingham, D. B. Chelton, D. B. Mann, and H. P. Hernandez, ASTM Bull. No. 240, 34 (TP164) 1959.

A porcelain reference electrode conductive to sodium ions for use in molten salt systems, R. J. Labrie and V. A. Lamb, J. Electrochem. Soc. **106,** No. 10, 895 (1959).

Memorandum on a procedure for obtaining spectral radiant intensities of tungsten-filament lamps, 400–700 mμ, L. E. Barbrow, J. Opt. Soc. Am. Letter to Editor **49,** No. 11, 1122 (1959).

Paramagnetic resonance spectra of active species. Blue material from hydrazoic acid, W. B. Gager and F. O. Rice, J. Chem. Phys. **31,** No. 2, 564 (1959).

Ceramic wafer tubes for modular units, C. P. Marsden, Section on Components and Materials, Electronics **32,** No. 39, 94 (1959).

Magnified and squared VSWR responses for microwave reflection coefficient measurements, R. W. Beatty, IRE Trans. on Microw. Theory and Tech. **MTT–7,** No. 3, 346 (1959).

Argon deposition on a 4.2° K surface, S. N. Foner, F. A. Mauer, and L. H. Bolz, J. Chem. Phys. **31,** No. 2, 546 (1959).

Low-temperature tensile properties of copper and four bronzes, R. M. McClintock, D. A. Van Gundy, and R. H. Kropschot, ASTM Bull, No. 240, 34 (TP177) 1959.

A nomenclature for conformations of pyranoid sugars and derivatives, H. S. Isbell and R. S. Tipson, Science **130,** No. 3378, 793 (1959).

On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. III (General Rayleigh quotient and characteristic roots with linear elementary divisors) A. M. Ostrowski, Arch. Rat. Mech. Anal. **3,** 325 (1959).

On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. IV (Generalized Rayleigh quotient for nonlinear elementary divisors) A. M. Ostrowski, IV Arch. Rat. Mech. Anal. **3,** 341 (1959).

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